DISCRETE MATHEMATICS AND ITS APPLICATIONS

COMBINATORICS

Second Edition

Nicholas A. Loehr

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Mathematics

DISCRETE MATHEMATICS AND ITS APPLICATIONS

Combinatorics, Second Edition is a well-rounded, general introduction to the subjects of enumerative, bijective, and algebraic combinatorics. The textbook emphasizes bijective proofs, which provide elegant solutions to counting problems by setting up one-to-one correspondences between two sets of combinatorial objects. The author has written the textbook to be accessible to readers without any prior background in abstract algebra or combinatorics.

Part I of the second edition develops an array of mathematical tools to solve counting problems: basic counting rules, recursions, inclusion-exclusion techniques, generating functions, bijective proofs, and linear algebraic methods. These tools are used to analyze combinatorial structures such as words, permutations, subsets, functions, graphs, trees, lattice paths, and much more.

Part II cover topics in algebraic combinatorics including group actions, permutation statistics, symmetric functions, and tableau combinatorics.

This edition provides greater coverage of the use of ordinary and exponential generating functions as a problem-solving tool. Along with two new chapters, several new sections, and improved exposition throughout, the textbook is brimming with many examples and exercises of various levels of difficulty.

Features

- This second edition presents a rigorous, yet readable exposition of enumerative and algebraic combinatorics
- Places an emphasis on bijective methods and combinatorial proofs
- Features two new chapters and several new sections
- Contains many examples and applications, along with over 1,000 exercises ranging in difficulty from routine verifications to unsolved problems

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Second Edition

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Dedication

This book is dedicated to

my sister Heather (1973–2015),
my aunt Nanette (1963–2011),
and my grandmother Oliva (1926–1982).

Preface to the Second Edition

This book presents a general introduction to enumerative, bijective, and algebraic combinatorics. *Enumerative combinatorics* is the mathematical theory of counting. This branch of discrete mathematics has flourished in the last few decades due to its many applications to probability, computer science, engineering, physics, and other areas. *Bijective combinatorics* produces elegant solutions to counting problems by setting up one-to-one correspondences (bijections) between two sets of combinatorial objects. *Algebraic combinatorics* uses combinatorial methods to obtain information about algebraic structures such as permutations, polynomials, matrices, and groups. This relatively new subfield of combinatorics has had a profound influence on classical mathematical subjects such as representation theory and algebraic geometry.

Part I of the text covers fundamental counting tools including the Sum and Product Rules, binomial coefficients, recursions, bijective proofs of combinatorial identities, enumeration problems in graph theory, inclusion-exclusion formulas, generating functions, ranking algorithms, and successor algorithms. This part requires minimal mathematical prerequisites and could be used for a one-semester combinatorics course at the advanced undergraduate or beginning graduate level. This material will be interesting and useful for computer scientists, statisticians, engineers, and physicists, as well as mathematicians.

Part II of the text contains an introduction to algebraic combinatorics, discussing groups, group actions, permutation statistics, tableaux, symmetric polynomials, and formal power series. My presentation of symmetric polynomials is more combinatorial (and, I hope, more accessible) than the standard reference work [84]. In particular, a novel approach based on antisymmetric polynomials and abaci yields elementary combinatorial proofs of some advanced results such as the Pieri Rules and the Littlewood–Richardson Rule for multiplying Schur symmetric polynomials. Part II assumes a bit more mathematical sophistication on the reader's part (mainly some knowledge of linear algebra) and could be used for a one-semester course for graduate students in mathematics and related areas. Some relevant background material from abstract algebra and linear algebra is reviewed in an appendix. The final chapter consists of independent sections on optional topics that complement material in the main text. In many chapters, some of the harder material in later sections can be omitted without loss of continuity.

Compared to the first edition, this new edition has an earlier, expanded treatment of generating functions that focuses more on the combinatorics and applications of generating functions and less on the algebraic formalism of formal power series. In particular, we provide greater coverage of exponential generating functions and the use of generating functions to solve recursions, evaluate summations, and enumerate complex combinatorial structures. We cover successor algorithms in more detail in Chapter 6, providing automatic methods to create these algorithms directly from counting arguments based on the Sum and Product Rules. The final chapter contains some new material on quasisymmetric polynomials. Many chapters in Part I have been reorganized to start with elementary content most pertinent to solving applied problems, deferring formal proofs and advanced material until later. I hope this restructuring makes the second edition more readable and appealing than the first edition, without sacrificing mathematical rigor.

Each chapter ends with a summary, a set of exercises, and bibliographic notes. The book contains over 1200 exercises, ranging in difficulty from routine verifications to unsolved problems. Although we provide references to the literature for some of the major theorems and harder problems, no attempt has been made to pinpoint the original source for every result appearing in the text and exercises.

I am grateful to the editors, reviewers, and other staff at CRC Press for their help with the preparation of this second edition. Readers may communicate errors and other comments to the author by sending e-mail to `nloehr@vt.edu`.

Nicholas A. Loehr

Introduction

The goal of *enumerative combinatorics* is to count the number of objects in a given finite set. This may seem like a simple task, but the sets we want to count are often very large and complicated. Here are some examples of enumeration problems that can be solved using the techniques in this book. How many encryption keys are available using the 128-bit AES encryption algorithm? How many strands of DNA can be built using five copies of each of the nucleotides adenine, cytosine, guanine, and thymine? How many ways can we be dealt a full house in five-card poker? How many ways can we place five rooks on a chessboard with no two rooks in the same row or column? How many subsets of $\{1, 2, \ldots, n\}$ contain no two consecutive integers? How many ways can we write 100 as a sum of positive integers? How many connected graphs on n vertices have no cycles? How many integers between 1 and n are relatively prime to n? How many circular necklaces can be made with three rubies, two emeralds, and one diamond if all rotations of a given necklace are considered the same? How many ways can we tile a chessboard with dominos? The answers to such counting questions can help us solve a wide variety of problems in probability, cryptography, algorithm analysis, physics, abstract algebra, and other areas of mathematics.

Part I of this book develops the basic principles of counting, placing particular emphasis on the role of *bijections*. To give a *bijective proof* that a given set S has size n, one must construct an explicit one-to-one correspondence (bijection) from S onto the set $\{1, 2, \ldots, n\}$. More generally, one can prove that two sets A and B have the same size by exhibiting a bijection between A and B. For example, given a fixed positive integer n, let A be the set of all strings $w_1w_2 \cdots w_{2n}$ consisting of n left parentheses and n right parentheses that are *balanced* (every left parenthesis can be matched to a right parenthesis later in the sequence). Let B be the set of all arrays

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix}$$

such that every number in $\{1, 2, \ldots, 2n\}$ appears once in the array, $y_1 < y_2 < \cdots < y_n$, $z_1 < z_2 < \cdots < z_n$, and $y_i < z_i$ for every i. The sets A and B seem quite different at first glance. Yet, we can demonstrate that A and B have the same size using the following bijection. Given $w = w_1w_2 \cdots w_{2n}$ in A, let $y_1, y_2, \ldots, y_n$ be the positions of the left parentheses in w (written in increasing order), and let $z_1, z_2, \ldots, z_n$ be the positions of the right parentheses in w (written in increasing order). For example, the string `(()())((()))()` in A maps to the array

$$\begin{bmatrix} 1 & 2 & 4 & 7 & 8 & 9 & 13 \\ 3 & 5 & 6 & 10 & 11 & 12 & 14 \end{bmatrix}.$$

One may check that the requirement $y_i < z_i$ for all i is equivalent to the fact that w is a *balanced* string of parentheses. The string w is uniquely determined by the array of y_i's and z_i's, and every such array arises from some string w in A. Thus we have defined the required one-to-one correspondence between A and B. We now know that the sets A and B have the same size, although we have not yet determined what that size is!

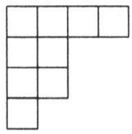
Bijective proofs, while elegant, can be very difficult to discover. For example, let C be the set of rearrangements of $1, 2, \ldots, n$ that have no decreasing subsequence of length three. It turns out that the sets B and C have the same size, so there must exist a bijection from

B to C. Can you find one? (Before spending too long on this question, you might want to read §12.13.)

Luckily, the field of enumerative combinatorics contains a whole arsenal of techniques to help us solve complicated counting problems. Besides bijections, some of these techniques include recursions, generating functions, group actions, inclusion-exclusion formulas, linear algebra, probabilistic methods, and symmetric polynomials. In the rest of this introduction, we describe several challenging enumeration problems that can be solved using these more advanced methods. These problems, and the combinatorial technology needed to solve them, will be discussed at greater length later in the text.

Standard Tableaux

Suppose we are given a diagram D consisting of a number of rows of boxes, left-justified, with each row no longer than the one above it. For example, consider this diagram:



Let n be the total number of boxes in the diagram. A *standard tableau of shape D* is a filling of the boxes in D with the numbers $1, 2, \ldots, n$ (used once each) so that every row forms an increasing sequence (reading left to right), and every column forms an increasing sequence (reading top to bottom). For example, here are three standard tableaux of shape D, where D is the diagram pictured above:

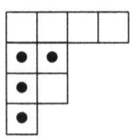
1	2	3	4
5	6		
7	8		
9			

1	3	4	9
2	5		
6	7		
8			

1	2	3	7
4	8		
5	9		
6			

Question: *Given a diagram D of n cells, how many standard tableaux of shape D are there?*

There is a remarkable answer to this counting problem, known as the *Hook-Length Formula.* To state it, we need to define hooks and hook lengths. The *hook* of a box b in a diagram D consists of all boxes to the right of b in its row, all boxes below b in its column, and box b itself. The *hook length* of b, denoted $h(b)$, is the number of boxes in the hook of b. For example, if b is the first box in the second row of D, then the hook of b consists of the marked boxes in the following picture:



So $h(b) = 4$. In the picture below, we have labeled each box in D with its hook length.

7	5	2	1
4	2		
3	1		
1			

Hook-Length Formula: *Given a diagram D of n cells, the number of standard tableaux of shape D is n! divided by the product of the hook lengths of all the boxes in D.*