MATHEMATICAL METHODS IN THE PHYSICAL SCIENCES

Second Edition

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PREFACE

This book is particularly intended for the student with one year of calculus who wants to develop, in a short time, a basic competence in each of the many areas of mathematics needed in junior to senior-graduate courses in physics, chemistry, and engineering. Thus it is intended to be accessible to sophomores (or freshmen with AP calculus from high school). It may also be used effectively by a more advanced student to review half-forgotten topics and learn new ones, either in independent study or in a class. Although the book was written especially for students of the physical sciences, students in any field (say, mathematics or mathematics for teaching) may find it useful to survey many topics or to obtain some knowledge of areas they do not have time to study in depth. Since theorems are stated carefully, such students should not have to unlearn anything in their later work.

The question of proper mathematical training for students in the physical sciences is of concern to both mathematicians and those who use mathematics in applications. Mathematicians are apt to claim that if students are going to study mathematics at all, they should study it in careful and thorough detail. For the undergraduate physics, chemistry, or engineering student, this means either (1) learning more mathematics than a mathematics major or (2) learning a few areas of mathematics thoroughly and the others only from snatches in science courses. The second alternative is often advocated; let me say why I think it is unsatisfactory. It is certainly true that motivation is increased by the immediate application of a mathematical technique, but there are a number of disadvantages.

1. The discussion of the mathematics is apt to be sketchy since that is not the primary concern.

2. Students are faced simultaneously with learning a new mathematical method and applying it to an area of science that is also new to them. Frequently the difficulty in comprehending the new scientific area lies more in the distraction caused by poorly understood mathematics than it does in the new scientific ideas.

3. Students may meet what is actually the same mathematical principle in two different science courses without recognizing the connection, or even learn apparently contradictory theorems in the two courses! For example, in thermodynamics students learn that the integral of an exact differential around a closed path is always zero. In electricity or hydrodynamics, they run into $\int_0^{2\pi} d\theta$, which is certainly the integral of an exact differential around a closed path but is not zero!

Now it would be fine if every science student could take the separate mathematics courses in differential equations (ordinary and partial), advanced calculus, linear algebra, vector and tensor analysis, complex variable, Fourier series, probability, calculus of variations, special functions, and so on. However, most science students have neither the time nor the inclination to study that much mathematics, yet they are constantly hampered in their science courses for lack of the basic techniques of these subjects. It is the intent of this book to give these students enough background in each of the needed areas so that they can cope successfully with junior, senior, and beginning graduate courses in the physical sciences. I hope, also, that some students will be sufficiently intrigued by one or more of the fields of mathematics to pursue it further.

It is clear that something must be omitted if so many topics are to be compressed into one course. I believe that two things can be left out without serious harm at this stage of a student's work: generality and detailed proofs. Stating and proving a theorem in its most general form is important to the mathematician and to the advanced student, but it is often unnecessary and may be confusing to the more elementary student. This is not in the least to say that the science student has no use for careful mathematics. Scientists, even more than mathematicians, need careful statements of the limits of applicability of mathematical processes so that they can use them with confidence without having to supply proof of their validity. Consequently I have endeavored to give accurate statements of the needed theorems, although often for special cases or without proof. Interested students can easily find more detail in textbooks in the special fields; see References (pp. 741ff.).

Other texts on this subject almost invariably assume a degree of mathematical sophistication and knowledge about advanced physics not yet attained by students at the sophomore level. Yet such students, if given simple and clear explanations, can master these techniques in rapid succession. (They not only *can*, but will *have to* one way or another, if they are going to pass their junior and senior physics courses!) These students are not ready for detailed applications—these they will get in their science courses—but they do need and want to be given some idea of the use of the methods and some simple applications. This I have tried to do for each new topic. Many other texts are slanted toward some particular field and contain a good many chapters on advanced topics in that special field. It is the purpose of this text to cover in simple fashion just the basic methods.

During the 17 years since the publication of the first edition of this book, I have become more than ever convinced of the value of offering this material at the sophomore level to students in the physical sciences. Without it, they must struggle to learn about partial differential equations, special functions, Fourier series and transforms, probability, calculus of variations, diagonalization of matrices, etc., as they meet these topics in their junior-senior courses in quantum mechanics, electrodynamics, classical mechanics, optics, etc. Thus the basic character of this text is unchanged. The principal changes in the second edition are as follows.

1. Chapter 3 has been completely rewritten to include the row reduction method of solving sets of simultaneous equations and inverting matrices and vector algebra has been moved to this chapter. However, I disagree strongly with those who would neglect determinants, Cramer's rule, and the formula for the inverse of a matrix, in favor of row reduction. In spite of the importance of the computer, mathematical physics is much more than arithmetic, and formulas are indispensable in theoretical physics. Thus I have presented both computational and formula methods and have indicated their importance for different purposes.
2. The number of problems has been approximately doubled. Students need to solve many problems in order to master this material and learn good problem-solving techniques. Appropriate problems are now included in the individual sections; also the last section of each chapter is a set of miscellaneous problems. Note the problem numbering system: Problem 7.2 means Problem 2 of Section 7; however, within Section 7, it is referred to as just Problem 2. Also note that equation numbers are in parentheses, but problem numbers are not; thus (7.2) means equation (7.2).
3. Important formulas, definitions, theorems, etc., have been boxed to make them easier to find.
4. The old Chapter 4 (Partial differentiation and multiple integrals) has been split into two chapters (4 and 5). In the new Chapter 5 (Multiple integrals), I have added introductory material on double and triple integrals, and have moved surface integrals (sec γ method) to this chapter. Some of the later chapters have been rearranged. Also new sections have been added on Hermite and Laguerre functions (Chapter 12, Section 22) and on Green functions (Chapter 15, Section 8).

The material in the text is so arranged that students who study the chapters in order through the book will have the necessary background at each stage. However, it is not always either necessary or desirable to follow the text order. Let me suggest some rearrangements I have found useful. With very little difficulty, it is possible to start with Chapter 3 or Chapter 4 or Chapter 5 instead of Chapter 1. If Chapter 4 precedes Chapter 1, the section on two-variable power series can be omitted until later. Assuming the class knows about vector algebra and determinants from Chapter 3, and a little about partial differentiation from Chapter 4, there is no trouble starting with Chapter 6. This provides the motivation of many physical applications. The class should return to Chapters 1 and 2 before going on to Chapters 7 and 8. If students have previously studied the material in any of Chapters 1, 3, 4, 5, 6, or 8 (in such courses as second-year calculus, differential equations, linear algebra), then the corresponding chapter(s) could be omitted, used for reference, or, preferably, be reviewed briefly with emphasis on problem solving. Students may know Taylor's theorem, for example, but be unable to find quickly and skillfully the Maclaurin series for $x^{-1} \sin(x^2)$; they may know the theory of multiple integrals, but find it difficult to set up a double integral for the moment of inertia of a spherical shell; they may know existence theorems for differential equations, but have little skill in solving, say, $y'' + y = x \sin x$.

After Chapter 7 (Fourier series) I like to cover the first four sections of Chapter 13 (partial differential equations) and the first four sections of Chapter 15 (Laplace and Fourier transforms). Although the later sections of Chapters 13 and 15 require more background, these early sections work out well following Chapter 7. Somewhat later in the course, we complete Chapter 13 after Chapter 12, and Chapter 15 after Chapter 14. Chapter 16 is almost independent of the rest of the text; I have covered this material anywhere from the very beginning to the very end of a one-year course. The following background is used in the other chapters.

For Chapter 9: differential equations (Chapter 8).

For Chapter 10: matrices (Chapter 3); vectors (Chapters 3 and 6); Lagrange's equations (Chapter 9).

For Chapter 11: power series (Chapter 1); differentiating integrals (Chapter 4); Lagrange's equations (Chapter 9), used in one example.

For Chapter 12: power series (Chapter 1); complex numbers (Chapter 2); partial differentiation (Chapter 4); differential equations (Chapter 8); gamma function (Chapter 11); background of Chapters 3, 6, 7, 10, used in discussion of eigenvalues and of orthogonal functions.

For Chapter 14: power series (Chapter 1); complex numbers (Chapter 2); partial differentiation (Chapter 4); vectors and vector theorems (Chapter 6); transformations (Chapter 10), used in one discussion.

I would like to suggest a number of different courses that can be given using the material in this text.

1. A one-year course in mathematical methods for students with a background of one (or one and a half) years of calculus. With these students, I usually cover about 13 or 14 of the 16 chapters in a one-year course meeting three times a week.
2. A one-semester or one-quarter course for students with more background (either advanced undergraduates or graduate students needing a brush-up course in the more advanced topics). For such students, I would start with Chapter 9 (or any desired later chapter) and refer back to earlier chapters as needed.
3. Differential equations. For this course there is a wealth of material greater than in most differential equations texts. I have used Chapter 8 as basic material with Chapter 2 for reference, and then, for the better students, material from Chapters 7, 12, 13, and 15 for series solutions, use of Laplace transforms, and an introduction to partial differential equations.
4. Fourier series and boundary value problems. There is ample material in Chapters 7, 12, and 13 for this course.

Please note two useful items at the end of the book: References and Bibliography (pp. 741ff.) and Answers to Selected Problems (pp. 747ff.).

I want to thank my students over the years and also other readers for many helpful suggestions, and I welcome any further comments. I very much appreciate the many enthusiastic responses to the first edition, and I hope that the second may prove even more useful.

January 1983 Mary L. Boas

TO THE STUDENT

As you start each topic in this book, you will no doubt wonder and ask "Just why should I study this subject and what use does it have in applications?" There is a story about a young mathematics instructor who asked an older professor "What do you say when students ask about the practical applications of some mathematical topic?" The experienced professor said "I tell them!" This text tries to follow that advice. However, you must on your part be reasonable in your request. It is not possible in one book or course to cover both the mathematical methods and very many detailed applications of them. You will have to be content with some information as to the areas of application of each topic and some of the simpler applications. In your later courses, you will then use these techniques in more advanced applications.

One point about your study of this material cannot be emphasized too strongly: To use mathematics effectively in applications, you need not just knowledge, but *skill*. Skill can be obtained only through practice. You can obtain a certain superficial *knowledge* of mathematics by listening to lectures, but you cannot obtain *skill* this way. How many students have I heard say "It looks so easy when you do it," or "I understand it but I can't do the problems!" Such statements show lack of practice and consequent lack of skill. The only way to develop the skill necessary to use this material in your later courses is to practice by solving many problems. Always study with pencil and paper at hand. Don't just read through a solved problem—try to do it yourself! Then solve some similar ones from the problem set for that section trying to choose the most appropriate method from the solved examples. See Answers to Selected Problems (pp. 747ff.) and check your answers to any problems listed there. If you meet an unfamiliar term, look for it in the Index (pp. 775ff.) or in a dictionary if it is nontechnical.

My students tell me that one of my most frequent comments to them is "You're working too hard." There is no merit in spending hours producing a many-page solution to a problem that can be done by a better method in a few lines. Please ignore anyone who disparages problem-solving techniques as "tricks" or "shortcuts." You will find that the more able you are to choose effective methods of solving problems in your science courses, the easier it will be for you to master new material. But this means practice, practice, practice! The *only* way to learn to solve problems is to solve problems. In this text, you will find both drill problems and harder, more challenging problems. You should not feel satisfied with your study of a chapter until you can solve a reasonable number of these problems.

M.L.B.

CONTENTS

1

INFINITE SERIES, POWER SERIES

1. THE GEOMETRIC SERIES

As a simple example of many of the ideas involved in series, we are going to consider the geometric series. You may recall that in a geometric progression we multiply each term by some fixed number to get the next term. For example,

$$(1.1)\qquad \begin{aligned} &(a)\quad 2, 4, 8, 16, 32, \ldots,\\ &(b)\quad 1, \tfrac{2}{3}, \tfrac{4}{9}, \tfrac{8}{27}, \tfrac{16}{81}, \ldots,\\ &(c)\quad a, ar, ar^2, ar^3, \ldots, \end{aligned}$$

are geometric progressions. It is easy to think of examples of such progressions. Suppose the number of bacteria in a culture doubles every hour. Then the terms of (1.1a) represent the number by which the bacteria population has been multiplied after 1 hr, 2 hr, and so on. Or suppose a bouncing ball rises each time to $\frac{2}{3}$ of the height of the previous bounce. Then (1.1b) would represent the heights of the successive bounces in yards if the ball is originally dropped from a height of 1 yd.

In our first example it is clear that the bacteria population would increase without limit as time went on (mathematically, anyway; that is, assuming that nothing like lack of food prevented the assumed doubling each hour). In the second example, however, the height of bounce of the ball decreases with successive bounces, and we might ask for the total distance the ball goes. The ball falls a distance 1 yd, rises a distance $\frac{2}{3}$ yd and falls a distance $\frac{2}{3}$ yd, rises a distance $\frac{4}{9}$ yd and falls a distance $\frac{4}{9}$ yd, and so on. Thus it seems reasonable to write the following expression for the total distance the ball goes:*

$$(1.2)\qquad 1 + 2\cdot\tfrac{2}{3} + 2\cdot\tfrac{4}{9} + 2\cdot\tfrac{8}{27} + \cdots = 1 + 2\left(\tfrac{2}{3} + \tfrac{4}{9} + \tfrac{8}{27} + \cdots\right),$$

where the three dots mean that the terms continue as they have started (each one being $\frac{2}{3}$ the preceding one), and there is never a last term. Let us consider the expression in parentheses in (1.2), namely

$$\frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots. \tag{1.3}$$

This expression is an example of an *infinite series*, and we are asked to find its sum. Not all infinite series have sums; you can see that the series formed by adding the terms in (1.1a) does not have a finite sum. However, even when an infinite series does have a finite sum, we cannot find it by adding the terms because no matter how many we add there are always more. Thus we must find another method. (It is actually deeper than this; what we really have to do is to *define* what we mean by the sum of the series.)

Let us first find the sum of n terms in (1.3). The formula (Problem 2) for the sum of n terms of the geometric progression (1.1c) is

$$S_n = \frac{a(1 - r^n)}{1 - r}. \tag{1.4}$$

Using (1.4) in (1.3), we find

$$S_n = \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}[1 - (\frac{2}{3})^n]}{1 - \frac{2}{3}} = 2[1 - (\tfrac{2}{3})^n]. \tag{1.5}$$

As n increases, $(\frac{2}{3})^n$ decreases and approaches zero. Then the sum of n terms approaches 2 as n increases, and we say that the sum of the series is 2. (This is really a definition: The sum of an infinite series is the limit of the sum of n terms as $n \to \infty$.) Then from (1.2), the total distance traveled by the ball is $1 + 2 \cdot 2 = 5$. This is an answer to a mathematical problem. A physicist might well object that a bounce the size of an atom is nonsense! However, the infinite number of small terms of the series after a number of bounces, contribute very little to the final answer (see Problem 1). Thus it makes little difference (in our answer for the total distance) whether we insist that the ball rolls after a certain number of bounces or whether we include the entire series, and it is easier to find the sum of the series than to find the sum of, say, twenty terms.

Series such as (1.3) whose terms form a geometric progression are called *geometric series*. We can write a geometric series in the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots. \tag{1.6}$$

The sum of the geometric series (if it has one) is by definition

$$S = \lim_{n \to \infty} S_n, \tag{1.7}$$

where S_n is the sum of n terms of the series. By following the method of the example above, you can show (Problem 2) that a geometric series has a sum if and only if $|r| < 1$, and in this case the sum is