Calculus

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Preface

At the London School of Economics, a whole spectrum of courses on mathematical methods is given ranging from very elementary courses for those who know virtually no mathematics to courses for those researching in mathematical economics and the like. It is our intention to produce a number of books which cover the material in these courses. Each of the books will be an independent entity but it is our hope that the whole will prove greater than the sum of its parts.

The current book is a 'second-level' wo k on calculus. The main topic is the calculus of several variables but elementary differential and difference equations are also treated at some length. The emphasis is very strongly on 'how to do it' aspects of these topics rather than their theoretical basis. However, there is little point in learning formulae by rote (except in so far as this helps in passing examinations set by rote). To use a technique in practice it is necessary to have some understanding of why it works. We therefore supplement the description of the various techniques with brief explanations of their theoretical background. But formal proofs are never attempted and, wherever possible, geometrical arguments are used.

At the end of each chapter, a variety of applications are given. These are drawn from economics, statistics and operational research reflecting our interests at LSE. However, the mathematical techniques described in the book are, of course, far more widely applicable and we hope that the book will be found useful not only by those studying mathematics with a view to applications in the social sciences but also by physical scientists and engineers. Returning to the applications given at the end of each chapter, these are quite advanced compared with the general level of the text. The aim has been to generate some excitement about the potentialities of the mathematical techniques rather than to usurp the role of those teaching applied courses. We have therefore included material on such topics as the duality theorem of linear programming, the Kuhn–Tucker theorem, the Slutsky equations, the cobweb model and a wide range of statistical topics including the central limit theorem and the gambler's ruin problem. Students who find the mathematical techniques described in the main body of the text troublesome to grasp would be advised not to try and puzzle out the details of these applications but to come back to them when they are encountered again at a later stage. Scientists and engineers, incidentally, might well benefit from observing the scope of these applications in the social sciences.

It is assumed that readers of this book will have some previous knowledge of the calculus of functions of one real variable. It is quite unsuitable for those with no knowledge whatsoever of this subject. On the other hand, the material treated in this book is taught at LSE to an exceedingly disparate group of students from all over the world. Some of these know very little. Others are graduate students brushing up their knowledge. We have therefore found it necessary to provide a substantial amount of revision material on topics which it would be better for readers to know properly before starting on this book. It is surprising how large the holes can be in the knowledge even of those whose previous mathematical education is entirely orthodox. Our experience has been that it is unprofitable to place this revision material on the calculus of one variable in a block at the beginning of the book. The temptation to neglect it altogether is then almost irresistible. It has therefore been somewhat slyly interwoven with the main body of the text in the hope that all readers will at least skim through it before moving on to new topics.

Some knowledge of linear algebra is also assumed. However, the level of understanding required is not a high one and sections explaining the basic ideas are included where appropriate. But it should be appreciated that these brief passages are not intended as a substitute for a course in linear algebra. (At LSE, students take a concurrent course in linear algebra while studying the material covered in this book.) As in the case of the calculus of one variable, those who are totally ignorant of the subject should begin with a more elementary book than this.

Sections which are intended as revision material and hence survey the ideas covered rather than explain them are marked with the symbol ¤. These sections should at the very least be scanned quickly to make sure the notation and techniques are all familiar. Certain other sections are marked with the symbol †. These should be omitted altogether by those who find the text difficult to cope with.

Finally, attention should be drawn to the examples and problems. When studying a 'how to do it' book, the criterion of success is whether or not one has learned 'how to do it'. Thus, a reader should count himself successful if and only if he is able to solve a substantial percentage of the problems which are given. This remains the case even for those who are not too sure whether they understand the foundations of the subject. When a formal subject like mathematics is presented informally as in this book, it is inevitable that all but those who are unusually gifted will have doubts about their grasp of the theory. Those who wish to dispel their doubts should consult a book in which the emphasis is on theoretical matters and proofs are given in a formal and precise way (e.g. the author's book *Mathematical Analysis: a straightforward approach*, also published by CUP). But, as far as the current book is concerned, a reader would be wise to accept that his understanding of the basic theory must be reasonably good if he can solve most of the problems since someone with little grasp of the theory would make no headway with the problems at all.

In any case, it is quite pointless to attempt to read this book without making a commitment to tackle the problems. Certainly far more time should be spent on attempting problems than on reading the text. To assist in this task, solutions are given at the end of the book to every other problem. Those for which no solution is given are marked with the symbol *. The usual (but not inevitable) pattern is that a starred question follows a rather similar unstarred question. In attempting a starred question it will therefore often be helpful to begin by first trying the preceding unstarred question and then consulting the solution given for this should this prove necessary. Obviously, however, little will be gained if the solution is consulted prematurely.

K. G. Binmore

Contents

1

Vectors and matrices

This book takes for granted that readers have some previous knowledge of the calculus of real functions of one real variable and also some knowledge of linear algebra. However, for those whose knowledge may be rusty from long disuse or raw with recent acquisition, sections on the necessary material from these subjects have been included where appropriate. Although these revision sections (marked with the symbol ¤) are as self-contained as possible, they are *not* suitable for those who are entirely ignorant of the topics covered. The material in the revision sections is surveyed rather than explained. It is suggested that readers who feel fairly confident of their mastery of this surveyed material scan through the revision sections quickly to check that the notation and techniques are all familiar before going on. Probably, however, there will be few readers who do not find something here and there in the revision sections which merits their close attention.

The current chapter is concerned with the fundamental techniques from linear algebra which we shall be using. This will be particularly useful for those who may be studying linear algebra concurrently with the present text.

Algebraists are sometimes neglectful of the geometric implications of their results. Since we shall be making much use of geometrical arguments, particular attention should therefore be paid to §1.16–§1.21, inclusive, in which the geometric relevance of various vector notions is explained. This material will be required almost immediately in chapter 2. The remaining material will not be needed until chapter 4. Those who are not very confident of their linear algebra may prefer leaving §1.31 until then.

1.1 Matrices¤

A matrix is a rectangular array of numbers. We usually enclose the array in brackets as in the examples below:

$$A = \begin{pmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

A matrix with m rows and n columns is called an $m \times n$ matrix. Thus A is a 3×2 matrix and B is a 2×3 matrix.

The numbers which appear in a matrix are called *scalars*. Sometimes it is useful to allow the scalars to be complex numbers but our scalars will always be *real numbers*.

1.2 Scalar multiplication⌑

One can do a certain amount of algebra with matrices and in this and the next few sections we shall describe the mechanics of some of the operations which are possible.

The first operation we shall consider is called *scalar multiplication*. If A is an $m \times n$ matrix and s is a scalar, then sA is the $m \times n$ matrix obtained by multiplying each entry of A by s. For example,

$$2A = 2\begin{pmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 4 & 2\times 1 \\ 2\times 0 & 2\times -1 \\ 2\times 3 & 2\times 2 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ 0 & -2 \\ 6 & 4 \end{pmatrix}.$$

Similarly,

$$5B = 5\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & -5 \\ 10 & 5 & 0 \end{pmatrix}.$$

1.3 Matrix addition and subtraction⌑

If C and D are two $m \times n$ matrices, then $C + D$ is the $m \times n$ matrix obtained by adding corresponding entries of C and D. Similarly, $C - D$ is the $m \times n$ matrix obtained by subtracting corresponding entries. For example, if

$$C = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 4 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 1 & 5 \\ -1 & -3 & 4 \\ -3 & 2 & 1 \end{pmatrix}$$

then

$$C+D=\begin{pmatrix}1+2 & -1+1 & 0+5\\ -2-1 & 3-3 & 1+4\\ 4-3 & 1+2 & 0+1\end{pmatrix}=\begin{pmatrix}3 & 0 & 5\\ -3 & 0 & 5\\ 1 & 3 & 1\end{pmatrix}$$

and

$$C-D=\begin{pmatrix}1-2 & -1-1 & 0-5\\ -2+1 & 3+3 & 1-4\\ 4+3 & 1-2 & 0-1\end{pmatrix}=\begin{pmatrix}-1 & -2 & -5\\ -1 & 6 & -3\\ 7 & -1 & -1\end{pmatrix}.$$

Note that

$$C+C=\begin{pmatrix}1+1 & -1-1 & 0+0\\ -2-2 & 3+3 & 1+1\\ 4+4 & 1+1 & 0+0\end{pmatrix}=\begin{pmatrix}2 & -2 & 0\\ -4 & 6 & 2\\ 8 & 2 & 0\end{pmatrix}=2C$$

and that

$$C-C=\begin{pmatrix}1-1 & -1+1 & 0-0\\ -2+2 & 3-3 & 1-1\\ 4-4 & 1-1 & 0-0\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The final matrix is called the 3×3 *zero matrix*. We usually denote any zero matrix by 0. This is a little naughty because of the possibility of confusion with other zero matrices or with the scalar 0. However, it has the advantage that we can then write

$$C-C=0$$

for any matrix C.

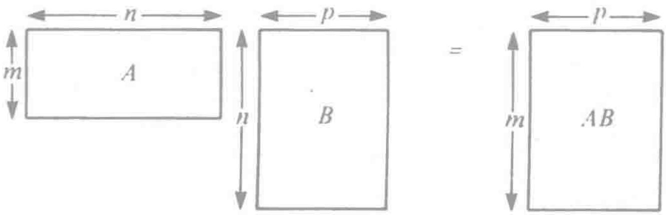
Note:
It makes *no* sense to try and add or subtract two matrices which are not of the same shape. Thus, for example,

$$A+B=\begin{pmatrix}4 & 1\\ 0 & -1\\ 3 & 2\end{pmatrix}+\begin{pmatrix}1 & 0 & -1\\ 2 & 1 & 0\end{pmatrix}$$

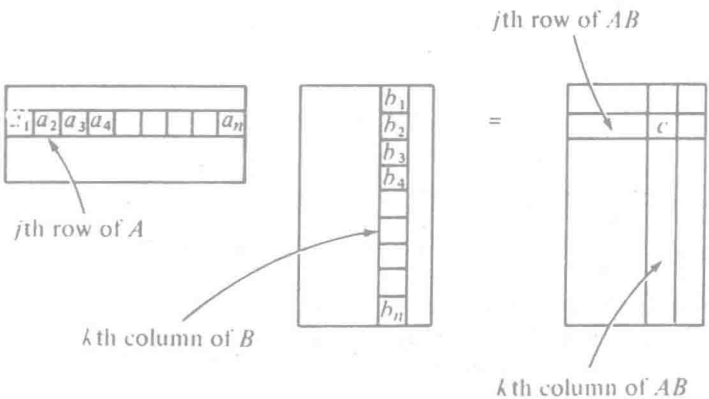
is an entirely *meaningless* expression.

1.4 Matrix multiplication[¤]

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then A and B can be multiplied to give an $m \times p$ matrix AB.



To work out the entry c of AB which appears in its jth row and kth column, we require the jth row of A and the kth column of B as illustrated below.



The entry c is then given by

$$c = a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n.$$

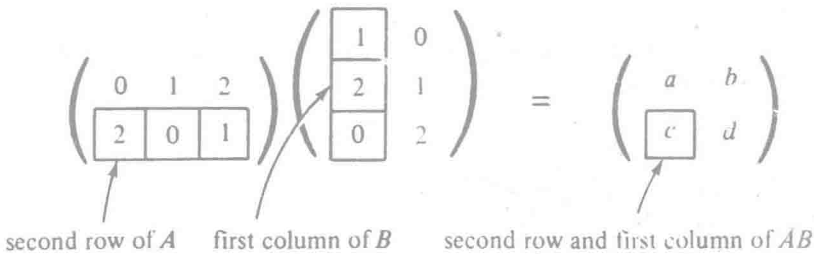
Example 1.5.[¤]
We compute the product AB of the matrices

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Since A is a 2×3 matrix and B is a 3×2 matrix, their product AB is a 2×2 matrix.

$$AB = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

To calculate c, we require the second row of A and the first column of B. These are indicated in the diagram below:



We obtain that

$$c = 2 \times 1 + 0 \times 2 + 1 \times 0 = 2 + 0 + 0 = 2.$$

Similarly,

$$a = 0 \times 1 + 1 \times 2 + 2 \times 0 = 0 + 2 + 0 = 2$$

$$b = 0 \times 0 + 1 \times 1 + 2 \times 2 = 0 + 1 + 4 = 5$$

$$d = 2 \times 0 + 0 \times 1 + 1 \times 2 = 0 + 0 + 2 = 2.$$

Thus

$$AB = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 2 & 2 \end{pmatrix}.$$

Note:

It makes *no* sense to try and calculate AB unless the number of columns in A is the same as the number of rows in B. Thus, for example,

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 3 \end{pmatrix}$$

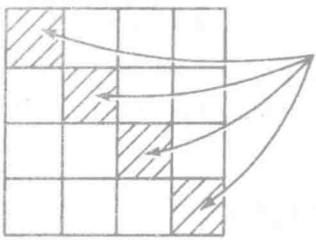
is an entirely *meaningless* expression.

1.6 Identity matrices

An $n \times n$ matrix is called a *square matrix* for obvious reasons. Thus, for example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

is a square matrix. The main *diagonal* of a square matrix is indicated in the diagram below:



The $n \times n$ *identity matrix* is the $n \times n$ matrix whose main diagonal entries are all 1 and whose other entries are all 0. We usually denote an identity matrix by I. The 3×3 identity matrix is

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that an identity matrix *must* be square. Just as a zero matrix behaves like the number 0, so an identity matrix behaves like the number 1. Specifically, we have that, if A is an $m \times n$ matrix, B is an $n \times p$ matrix and I is the $n \times n$ identity matrix, then

$$AI = A \quad \text{and} \quad IB = B.$$

Examples 1.7

(i) $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{pmatrix}.$

1.8 Determinants[¤]

With each *square* matrix there is associated a scalar called the *determinant* of the matrix. We shall denote the determinant of the square matrix A by $\det(A)$ or by $|A|$. (There is some risk of confusing the latter notation with the modulus or absolute value of a real number. Note that the determinant of a square matrix may be plus *or* minus.)

The general definition of a determinant is rather complicated and we therefore shall only explain how to calculate the determinants of 1×1, 2×2 and 3×3 matrices.

(i) 1×1 matrices. A 1×1 matrix $A = (a)$ is just a scalar and $\det(A) = a$.

(ii) 2×2 matrices. The determinant of the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is given by

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

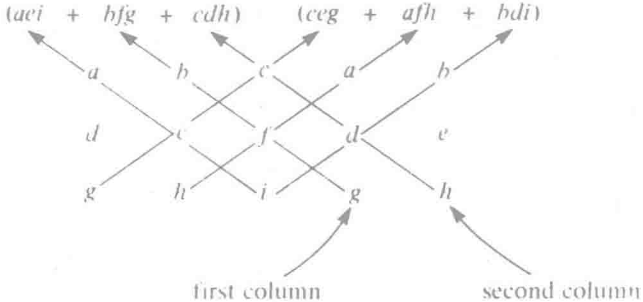
(iii) 3×3 matrices. The determinant of the 3×3 matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is given by

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (ceg + afh + bdi).$$

This is most easily remembered by drawing the diagram below:



Examples 1.9[¤]

(i) The determinant of the 1×1 matrix $A = (3)$ is simply $\det(A) = 3$.

(ii) The determinant of the 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

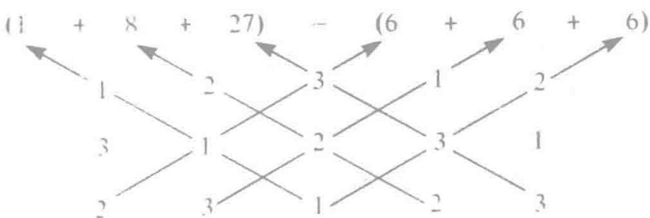
is

$$\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = 4 - 6 = -2.$$

(iii) We find the determinant of the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

using the diagram below:



Thus

$$\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (1+8+27)-(6+6+6) = 36-18 = 18$$

1.10 Inverse matrices¤

We have dealt with matrix addition, subtraction and multiplication and found that these operations only make sense in certain restricted circumstances. The circumstances under which it is possible to *divide* by a matrix are even more restricted.

A *non-singular* matrix is a *square* matrix whose determinant is *non-zero.* Each of the matrices of example 1.9 is therefore non-singular.

> Suppose that A is a matrix. Then there is another matrix B such that
>
> $$AB = BA = I$$
>
> if and only if A is non-singular.

In fact, if A is non-singular there is precisely *one* matrix B such that $AB = BA = I$. We call this matrix B the *inverse matrix* to A and write $B = A^{-1}$.

Thus a non-singular matrix A has an inverse matrix A^{-1} which satisfies

$$AA^{-1} = A^{-1}A = I.$$

If A is an $n \times n$ matrix, then A^{-1} is an $n \times n$ matrix as well (otherwise the equation above would make no sense).

If A is *not* square or if A is square but its determinant is *zero* (i.e. A is singular), then A does *not* have an inverse in the above sense.

If B is an $m \times n$ matrix and A is an $n \times n$ *non-singular* matrix, then one can define B/A by BA^{-1}. Note, however, that such a definition of division is severely restricted in its range of application.

1.11 Transpose matrices¤

In describing how to compute the inverse of a non-singular matrix, we shall need the idea of a transpose matrix. This is also useful in other connexions.

If A is an $m \times n$ matrix, then its *transpose* A^T is the $n \times m$ matrix whose first row is the first column of A, whose second row is the second column of A, whose third row is the third column of A and so on.

$$A = \underbrace{\begin{array}{|c|c|c|c|c|}\hline a_1 & b_1 & c_1 & d_1 & e_1 \\ \hline a_2 & b_2 & c_2 & d_2 & e_2 \\ \hline a_3 & b_3 & c_3 & d_3 & e_3 \\ \hline \end{array}}_{n}\ \Big\updownarrow m \qquad A^T = \underbrace{\begin{array}{|c|c|c|}\hline a_1 & a_2 & a_3 \\ \hline b_1 & b_2 & b_3 \\ \hline c_1 & c_2 & c_3 \\ \hline d_1 & d_2 & d_3 \\ \hline e_1 & e_2 & e_3 \\ \hline \end{array}}_{m}\ \Big\updownarrow n$$

Alternative notations for the transpose are A' or A^t.

An important special case is that when A is a square matrix for which $A = A^T$. Such a matrix is called *symmetric*.

Examples 1.12 □

(i) If $A = \begin{pmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}$, then $A^T = \begin{pmatrix} 4 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix}$.

Note that

$$(A^T)^T = \begin{pmatrix} 4 & 0 & 3 \\ 1 & -1 & 2 \end{pmatrix}^T = \begin{pmatrix} 4 & 1 \\ 0 & -1 \\ 3 & 2 \end{pmatrix} = A.$$

(ii) If $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & 0 \\ 5 & 0 & 4 \end{pmatrix}$, then $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & 0 \\ 5 & 0 & 4 \end{pmatrix}$.

Thus $A = A^T$ and so A is *symmetric*.

1.13 Cramer's rule □

Cramer's rule is a method for working out the inverse of a non-singular matrix. Other methods exist but Cramer's rule is usually easiest for 1×1, 2×2 and 3×3 matrices.

In the case of 1×1 and 2×2 matrices, one might as well learn the result of using Cramer's rule by heart.

(i) 1×1 matrices. A 1×1 non-singular matrix $A = (a)$ is just a non-zero scalar and