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INTRODUCTION TO FEYNMAN DIAGRAMS

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by

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FOREWORD

THIS book is based on lectures given to experimental physicists at the Joint Institute for Nuclear Research and students at the Dubna branch of Moscow State University. The lecture style in so far as possible has been preserved and it is for this reason that there are repetitions in the text.

The book presents Feynman diagram techniques and methods for calculating quantities measured experimentally (cross sections, polarizations). It is primarily intended for experimental physicists. The author hopes that the book will also be useful to advanced students specializing in elementary particle physics.

The graphical method of representing a perturbation theory series (the Feynman diagram) which appeared at the beginning of the 1950s has turned out to be extremely fruitful and is firmly established in all areas of physics. The significance of this method goes far beyond the boundaries of perturbation theory.

At the present time Feynman diagrams have become a widespread language which not only theoretical but also experimental physicists should know. Feynman diagrams are very intuitive; the rules for constructing them are simple. However, for a conscientious use of the Feynman diagram method we must proceed by introducing quantized fields, the S-matrix, and chronological and normal products.

The author's goal was to present all these topics as economically as possible. Proofs therefore have been simplified and only the minimum amount of rigor necessary for comprehension has been maintained.

There are examples of the most varied processes in which particles as well as antiparticles, identical particles, and so forth take part. The author tried to present methods for calculating the matrix elements of these processes (by perturbation theory) and the basic rules for constructing Feynman diagrams.

Finally, the author tried to present in detail the methods for calculating quantities measured experimentally. The techniques for calculating cross sections and polarizations is illustrated for a whole series of processes, each of which is examined thoroughly. The details of the calculations can easily be retraced by the reader.

The examples discussed here relate to elementary particle physics and include a wide range of processes with weak, electromagnetic, and strong interactions.

The processes in which both leptons and hadrons take part ($\nu_\mu + n \to \mu^- + p, e + p \to e + p$, and others) are examined in most detail. The electromagnetic and weak form factors of nucleons are discussed in great detail. The author, however, does not pretend to have provided any kind of orderly presentation of the present theory of weak and electromagnetic interactions.

A whole series of important problems in elementary particle physics are not examined in this book. We do not discuss here the analytic characteristics of the matrix elements (dispersion relations), higher orders of perturbation theory and renormalization theory, and much else. The presentation of these problems can be found in books on quantum field theory and elementary particle physics. These books contain further references to the original literature.

In conclusion I consider it my pleasant duty to thank Ya. A. Smorodinsky for useful discussions of the questions examined in the book, as well as D. Fakirov, N. M. Shumeyko, and especially L. L. Nemenov, who read the manuscript and made a number of significant comments.

CONTENTS

INTRODUCTION

BEFORE proceeding to the presentation of field theory we shall introduce the system of units generally accepted in relativistic quantum theory, in which Planck's constant $\hbar$ and the speed of light c are equal to one. Let us examine the relationship between this system and the cgs system. In the cgs system $\hbar$ and c are equal respectively to

$$\hbar = (\hbar)\,\frac{\mathrm{g}\times\mathrm{cm}^2}{\mathrm{sec}}, \qquad c = (c)\,\frac{\mathrm{cm}}{\mathrm{sec}},$$

where $(\hbar) \approx 1.05\times 10^{-27}$ and $(c) \approx 3\times 10^{10}$—the values for Planck's constant and the speed of light in the cgs system. Let us choose units of time and mass such that in the new system of units the numerical values of Planck's constant $\hbar$ and the speed of light c are equal to one. If we choose as units of time and mass

$$t_0 = \frac{\mathrm{sec}}{(c)}, \qquad m_0 = \frac{(\hbar)}{(c)}\,\mathrm{g},$$

then obviously in these units

$$\hbar = 1\,\frac{m_0\times\mathrm{cm}^2}{t_0}, \qquad c = 1\,\frac{\mathrm{cm}}{t_0}.$$

It is easy to obtain the relationships between the values of physical quantities in the cgs system and the new system of units. Denote the mass by m, the energy by E, and the momentum by p. Thus

$$m = (m)\,\mathrm{g} = (m)\frac{c}{\hbar}\,m_0 = (m)'\,m_0,$$

$$E = (E)\,\frac{\mathrm{g}\times\mathrm{cm}^2}{\mathrm{sec}^2} = (E)\,\frac{1}{(\hbar)\,(c)}\,\frac{m_0\times\mathrm{cm}^2}{t_0^2} = (E)'\,\frac{m_0\times\mathrm{cm}}{t_0},$$

$$p = (p)\,\frac{\mathrm{g}\times\mathrm{cm}}{\mathrm{sec}} = (p)\,\frac{1}{\hbar}\,\frac{m_0\times\mathrm{cm}}{t_0} = (p)'\,\frac{m_0\times\mathrm{cm}}{t_0}.$$

Here (m), (E), and (p) are the values for m, E, and p in the cgs system and $(m)'$, $(E)'$, and (p) are the values for the corresponding quantities in the system $\hbar = c = 1$. Thus we find

$$(m)' = (m)\,\frac{(c)}{(\hbar)}, \qquad (E)' = (E)\,\frac{1}{(\hbar)\,(c)}, \qquad (p)' = (p)\,\frac{1}{(\hbar)}.$$

In an analogous manner it is not difficult to obtain the relationships between the values of any physical quantities in the cgs system and in the system in which $\hbar = c = 1$.

Not all relationships between physical quantities in the new units contain the dimensional constants $\hbar$ and c (for example, the relativistic relationship between energy and momentum in these units takes the form $E^2 = m^2 + p^2$). This is equivalent to the fact that in the system where $\hbar = c = 1$, $TL^{-1} = 1$ and $ML^2T = 1$ and consequently the units of measurement of all quantities are expressed in terms of L. Obviously the mass has the dimension $M = L^{-2}T = L^{-1}$, the energy has the dimension L^{-1}, the angular momentum is a dimensionless quantity, the momentum has dimension $MLT^{-1} = L^{-1}$, and so forth.

Note that four-vectors will always be written in the form $A = (\boldsymbol{A}, iA_0)$. The square of this four-vector (a scalar) is $A^2 = \boldsymbol{A}^2 + A_4^2 = \boldsymbol{A}^2 - A_0^2$.

CHAPTER 1

THE *S*-MATRIX

1. The Interaction Representation. The *S*-matrix

Quantum field theory describes those particle interactions in which the number of particles may not be conserved. We begin by introducing the *S*-matrix, the matrix whose elements give the probability amplitudes for the corresponding transitions.

The basic postulate of quantum field theory is that the equation of motion is the Schrödinger equation:

$$i\frac{\partial\Psi(t)}{\partial t} = H\Psi(t). \tag{1.1}$$

Here $\Psi(t)$ is the wave function describing the system at time t and H is the full Hamiltonian of the system. If the system is closed, the Hamiltonian does not depend on time. It is further postulated that operators correspond to physical quantities in quantum field theory as in ordinary quantum mechanics and the mean values of the operators

$$\langle O\rangle = (\Psi^{+}O\Psi) \tag{1.2}$$

are the observed quantities. In (1.2) O is the operator corresponding to some physical quantity and Ψ is the wave function describing the state.

Let V be an arbitrary unitary operator:

$$V^{+}V = 1. \tag{1.3}$$

Then it is obvious that

$$\langle O\rangle = (\Psi'^{+}O'\Psi'), \tag{1.4}$$

where†

$$\Psi' = V\Psi, \quad \Psi'^{+} = \Psi^{+}V^{+}, \quad O' = VOV^{+}. \tag{1.5}$$

† We use matrix notation for the operators and functions. In components, equation (1.1) is written in the following manner:

$$i\frac{\partial\Psi_{\alpha}(t)}{\partial t} = \sum_{\alpha'} H_{\alpha\alpha'}\Psi_{\alpha'}(t).$$

The mean value of the operator O by definition is equal to $\langle O\rangle = \sum_{\alpha',\alpha} \Psi^{*}_{\alpha'}O_{\alpha'\alpha}\Psi_{\alpha}$. As a matrix, this expression can be written in the form (1.2). Obviously

$$\Psi'^{*}_{\alpha} = \sum_{\alpha'} V^{*}_{\alpha\alpha'}\Psi^{*}_{\alpha'} = \sum_{\alpha'} \Psi^{*}_{\alpha'}(V^{+})_{\alpha'\alpha} = (\Psi^{+}V^{+})_{\alpha}.$$

Relationships (1.4) and (1.5) imply that the mean value of the operator does not change if we use the wave function $\Psi' = V\Psi$ for describing the system instead of the wave function Ψ, while instead of the operator O, the operator $O' = VOV^+$ corresponds to the physical quantity. In other words, the mean values of the operators (quantities measured experimentally) are invariant under the unitary transformation (1.5).

The unitary operator V can be chosen such that the new wave functions change with time only when there is an interaction. Actually, V is an arbitrary unitary operator. It can also depend on time. Let us denote such an operator by $V(t)$ and the new functions and operators by

$$\Phi(t) = V(t)\,\Psi(t), \quad O(t) = V(t)OV^+(t). \tag{1.6}$$

We write the full Hamiltonian H in the form

$$H = H_0 + H_I, \tag{1.7}$$

where H_0 is the free Hamiltonian and H_I is the interaction Hamiltonian. Substituting $\Psi(t) = V^+(t)\Phi(t)$ in the equation of motion (1.1) we obtain

$$i\frac{\partial V^+(t)}{\partial t}\Phi(t) + iV^+(t)\frac{\partial \Phi(t)}{\partial t} = (H_0 + H_I)\,V^+(t)\Phi(t). \tag{1.8}$$

Now let us choose the unitary operator $V(t)$ such that

$$i\frac{\partial V^+(t)}{\partial t} = H_0 V^+(t). \tag{1.9}$$

Multiplying (1.8) by $V(t)$ and using (1.9) we obtain the equation

$$i\frac{\partial \Phi(t)}{\partial t} = H_I(t)\Phi(t), \tag{1.10}$$

where

$$H_I(t) = V(t)\,H_I V^+(t). \tag{1.11}$$

The transition from the description of the system with wave functions $\Psi(t)$ which satisfy Schrödinger's equation (1.1) to the description with functions $\Phi(t) = V(t)\,\Psi(t)$, where $V(t)$ is a unitary operator satisfying equation (1.9), is called the transition from the Schrödinger representation to the Dirac representation (interaction representation). The wave function of the system in the interaction representation, as can be seen from equation (1.10), depends on time only when there is an interaction. The interaction representation is widely used in quantum field theory and henceforth we, as a rule, shall work in this representation. Note that the general solution of equation (1.9) is

$$V^+(t) = \mathrm{e}^{-iH_0(t-t_1)}V^+(t_1), \tag{1.12}$$

where $V^+(t_1)\,V(t_1) = 1$. If we assume that the representations coincide at time t_1, then $V(t_1) = 1$ and

$$V^+(t) = \mathrm{e}^{-iH_0(t-t_1)}. \tag{1.13}$$

As can be seen from (1.6), the operators in the interaction representation depend on time in general. Differentiating $O(t)$ with respect to t and using the equation

$$-i\frac{\partial V(t)}{\partial t} = V(t)\,H_0, \tag{1.14}$$

obtained from (1.9) by Hermitian conjugation, we find

$$i\frac{\partial O(t)}{\partial t} = -V(t)H_0OV^+(t)+V(t)\,OH_0V^+(t) = [O(t), H_0(t)]. \tag{1.15}$$

We assumed for this that the operator O in the original Schrödinger representation does not depend on time. Relationship (1.15) defines the time-dependence of the operators in the interaction representation. From (1.15) it is obvious that the free Hamiltonian in this representation does not depend on time, i.e. $H_0(t) = H_0$.

Now let us determine the general solution of the equation of motion (1.10). For this we examine the equivalent integral equation

$$\Phi(t) = \Phi(t_0)+(-i)\int_{t_0}^{t} dt_1 H_I(t_1)\,\Phi(t_1), \tag{1.16}$$

where $\Phi(t_0)$ is the wave function of the system at initial time t_0. The initial wave function must be given. Our task consists in finding the wave function of the system at any subsequent time t. Substituting for $\Phi(t_1)$ under the integral sign on the right-hand side of (1.16) the sum

$$\Phi(t_0)+(-i)\int_{t_0}^{t_1} dt_2 H_I(t_2)\,\Phi(t_2),$$

we obtain

$$\Phi(t) = \Phi(t_0)+(-i)\int_{t_0}^{t} dt_1 H_I(t_1)\,\Phi(t_0)+(-i)^2\int_{t_0}^{t} dt_1 H_I(t_1)\int_{t_0}^{t_1} dt_2 H_I(t_2)\,\Phi(t_2). \tag{1.17}$$

Continuing this procedure we find

$$\Phi(t) = \left[1+(-i)\int_{t_0}^{t} dt_1 H_I(t_1)+(-i)^2\int_{t_0}^{t} dt_1 H_I(t_1)\int_{t_0}^{t_1} dt_2 H_I(t_2)+\ldots \right.$$
$$\left.+(-i)^n\int_{t_0}^{t} dt_1 H_I(t_1)\int_{t_0}^{t_1} dt_2 H_I(t_2)\ldots\int_{t_0}^{t_{n-1}} dt_n H_I(t_n)+\ldots\right]\Phi(t_0). \tag{1.18}$$

Thus, the general solution of the equation of motion (1.10) is obtained in the form of a series in powers of the interaction Hamiltonian. We are not going to discuss the question of the convergence of this series or the possibility of summing it. We shall only note that in the case when the interaction is characterized by a small constant, the first few members of the series (1.18) already can give the solution with sufficient accuracy. This occurs in the case of the electromagnetic interaction as a result of the smallness of the fine structure constant $\alpha = e^2/4\pi \approx 1/137$.

Let us write the solution (1.18) in the form

$$\Phi(t) = U(t,\ t_0)\,\Phi(t_0), \tag{1.19}$$

where the operator $U(t, t_0)$ is equal to

$$U(t, t_0) = \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 H_I(t_1) \int_{t_0}^{t_1} dt_2 H_I(t_2) \ldots \int_{t_0}^{t_{n-1}} dt_n H_I(t_n). \tag{1.20}$$

Thus, the solution of the equation of motion at time t can be obtained by operating with $U(t, t_0)$, which is defined by the interaction Hamiltonian, on the wave function given at initial time t_0.

It is not difficult to see that $U(t, t_0)$ is a unitary operator. Actually, it follows from (1.10) and (1.19) that the operator $U(t, t_0)$ satisfies the equation

$$i \frac{\partial U(t, t_0)}{\partial t} = H_I(t)\, U(t, t_0). \tag{1.21}$$

Hence, with the help of the Hermitian conjugate we have (H_I is a Hermitian operator)

$$-i \frac{\partial U^+(t, t_0)}{\partial t} = U^+(t, t_0) H_I(t). \tag{1.22}$$

Let us multiply the left-hand side of (1.21) by $U^+(t, t_0)$ and the right-hand side of (1.22) by $U(t, t_0)$ and subtract the second relationship from the first. We obtain

$$\frac{\partial}{\partial t} (U^+(t, t_0)\, U(t, t_0)) = 0. \tag{1.23}$$

Thus, the operator $U^+(t, t_0)\, U(t, t_0)$ does not depend on t. Obviously

$$U(t_0, t_0) = 1. \tag{1.24}$$

From (1.23) and (1.24) we conclude that

$$U^+(t, t_0)\, U(t, t_0) = 1. \tag{1.25}$$

Let us write the members of the series (1.20) in a more convenient form. We shall first examine the third member of the series. The expression $H_I(t_1)\, H_I(t_2)$ is integrated over the shaded area in Fig. 1. The integration is first carried out over t_2 from t_0 to t_1, and then over t_1 from t_0 to t. Let us integrate $H_I(t_1)\, H_I(t_2)$ over the same area, but first over t_1 (from t_2 to t) and then over t_2 (from t_0 to t). Assuming that the value of the integral does not depend on the order of integration, we find

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2) = \int_{t_0}^{t} dt_2 \int_{t_2}^{t} dt_1\, H_I(t_1)\, H_I(t_2) = \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\, H_I(t_2)\, H_I(t_1). \tag{1.26}$$

The last equation is obtained by a change of the integration variables ($t_1 \rightleftharpoons t_2$). Thus, we have

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2) = \tfrac{1}{2} \int_{t_0}^{t} dt_1 \left[\int_{t_0}^{t_1} dt_2\, H_I(t_1)\, H_I(t_2) + \int_{t_1}^{t} dt_2\, H_I(t_2)\, H_I(t_1) \right]. \tag{1.27}$$

This expression can be written more compactly. For this we introduce the *Dyson chronolog-*

ical operator P:

$$P(H_I(t_1)\,H_I(t_2)) = \begin{cases} H_I(t_1)\,H_I(t_2), & t_1 > t_2, \\ H_I(t_2)\,H_I(t_1), & t_2 > t_1. \end{cases} \tag{1.28}$$

The operator P acting on a product of time-dependent operators arranges them so that the time argument decreases from left to right. From (1.27) and (1.28) we obtain

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H_I(t_1)\,H_I(t_2) = \tfrac{1}{2} \int_{t_1}^{t} dt_1 \int_{t_0}^{t} dt_2\, P(H_I(t_1)\,H_I(t_2)). \tag{1.29}$$

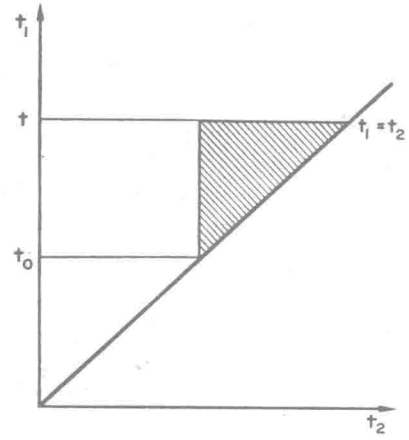

FIG. 1

We shall prove that in the general case

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t_1)\,H_I(t_2)\ldots H_I(t_n)$$

$$= \frac{1}{n!} \int_{t}^{t} dt_1 \int_{t_0}^{t} dt_2 \int_{t_0}^{t} dt_n\, P(H_I(t_1)\,H_I(t_2)\ldots H_I(t_n)), \tag{1.30}$$

where the Dyson chronological operator P in the case of n factors is defined such that

$$P(H_I(t_1)\,H_I(t_2)\ldots H_I(t_n)) = H_I(t_1)\,H_I(t_2)\ldots H_I(t_n) \quad \text{where} \quad t_1 > t_2 > \ldots > t_n. \tag{1.31}$$

The proof is by induction. Let relationship (1.30) be true for n factors. Let us examine the expression

$$I(t) = \int_{0}^{t} dt' \int_{t_0}^{t'} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n\, H_I(t')\,H_I(t_1)\,H_I(t_2)\ldots H_I(t_n)$$

$$- \frac{1}{(n+1)!} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n\, P(H_I(t')\,H_I(t_1)\,H_I(t_2)\ldots H_I(t_n)). \tag{1.32}$$

Differentiating $I(t)$ with respect to t we find

$$\frac{dI(t)}{dt} = H_I(t)\left[\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) H_I(t_2) \ldots H_I(t_n) - \frac{1}{n!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n P(H_I(t_1) H_I(t_2) \ldots H_I(t_n))\right]. \tag{1.33}$$

To obtain (1.33) the relationship

$$P(H_I(t_1) H_I(t_2) \ldots H_I(t_i) H_I(t) H_I(t_{i+1}) \ldots H_I(t_n)) = H_I(t) P(H_I(t_1) H_I(t_2) \ldots H_I(t_n)) \tag{1.34}$$

was used, which is obviously correct for arbitrary n ($t_1 \leqslant t$, $t_2 \leqslant t$, ...). If relationship (1.30) is true, then the right-hand side of (1.33) is equal to zero, i.e. I does not depend on time. Since $I(t_0) = 0$, this implies that $I(t) = 0$ for any t. Thus, from the validity of relationship (1.30) for n factors follows the validity for $n+1$ factors. Since we showed that relationship (1.30) is true for $n = 2$, it is thereby proved in general. Consequently, the operator $U(t, t_0)$ can be written in the form

$$U(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ldots \int_{t_0}^{t} dt_n P(H_I(t_1) \ldots H_I(t_n)). \tag{1.35}$$

Now we shall formulate the basic problem—the problem of the *collision of particles*. Let us assume that the initial state is given for $t_0 \to -\infty$; we are interested in the state of the system for $t \to \infty$. From (1.19) we obtain

$$\Phi(\infty) = U(\infty, -\infty)\Phi(-\infty). \tag{1.36}$$

Let us denote

$$U(\infty, -\infty) = S. \tag{1.37}$$

This operator is called the *S-matrix*.

Thus, the operator S acting on the initial wave function of the system given as $t \to -\infty$ gives the function of the system as $t \to \infty$. We have

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \ldots \int_{-\infty}^{\infty} dt_n P(H_I(t_1) H_I(t_2) \ldots H_I(t_n)). \tag{1.38}$$

From (1.25) and (1.37) it follows that the S-matrix is unitary, i.e.

$$S^+S = 1. \tag{1.39}$$

Let Φ_m be some complete orthonormalized set of functions. Expanding $\Phi(\infty)$ in functions Φ_m we obtain

$$\Phi(\infty) = \sum_m \Phi_m(\Phi_m^+ S\Phi(-\infty)). \tag{1.40}$$

Let us assume that at initial time the system is in the state Φ_n, i.e. $\Phi(-\infty) = \Phi_n$. Then

$$\Phi_n(\infty) = \sum_m \Phi_m(\Phi_m^+ S \Phi_n). \tag{1.41}$$

The matrix element $(\Phi_m^+ S \Phi_n) = S_{m;n}$ is therefore the *probability amplitude of the transition* from state Φ_n into state Φ_m. The complete set of functions Φ_m chosen in problems concerning the collision of particles is a set of functions which describe free particles with definite momenta. At the same time, in the general case particles can be different in the initial and final states. Besides that, in the process of colliding new particles may be produced. This type of process is described by quantum field theory. Below we shall give a brief account of the mathematical apparatus of quantum field theory and: (1) construct functions which describe free particles with definite momenta; (2) learn how to calculate (by perturbation theory) the matrix elements for the transition between such states; (3) examine in detail the techniques for calculating transition probabilities.

CHAPTER 2

CLASSICAL FIELDS

2. Equations of Motion

A classical system of N bodies is described by $3N$ functions of time $q_i(t)$ $(i = 1, \ldots, N)$, which satisfy Newton's equations:

$$m_i \ddot{q}_i = -\frac{\partial V}{\partial q_i}. \tag{2.1}$$

Here m_i is the mass of the ith particle and V is the potential energy.

As is known, the equations of motion (2.1) can be obtained from a variational principle. Let us recall its formulation. For the sake of simplicity we shall limit ourselves to the case of one-dimensional motion. Define the action

$$\mathcal{S} = \int_{t_0}^{t_1} L(q, \dot{q})\, dt, \tag{2.2}$$

where the Lagrange function $L(q, \dot{q})$ is equal to

$$L(q, \dot{q}) = \frac{m\dot{q}^2}{2} - V(q). \tag{2.3}$$

Obviously the value of the action $\mathcal{S}$ is determined by the function $q(t)$ over the interval from t_0 to t ($\mathcal{S}$ is a functional of $q(t)$). Besides $q(t)$ let us examine the function

$$q'(t) = q(t) + \delta q(t), \tag{2.4}$$

where $\delta q(t)$ is infinitesimal. Suppose

$$\delta q(t_0) = \delta q(t_1) = 0. \tag{2.5}$$

The increase of the action going from $q(t)$ to $q'(t)$ (the variation of the action) accurate to first order is equal to

$$\delta \mathcal{S} = \int_{t_0}^{t_1} [L(q', \dot{q}') - L(q, \dot{q})]\, dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q\, dt. \tag{2.6}$$

In obtaining (2.6) we used the fact that

$$\delta\dot{q} = \dot{q}' - \dot{q} = \frac{d}{dt}\delta q,$$

then integrated by parts and took (2.5) into account. We require that the variation of the action (2.6) vanish. Since $\delta q(t)$ is arbitrary, from (2.6) we find

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0. \tag{2.7}$$

Differentiating the Lagrangian function (2.3) with respect to q and $\dot{q}$, we obtain from (2.7)

$$m\ddot{q} = -\frac{\partial V}{\partial q},$$

i.e. (2.7) is the equation of motion. Thus, the function $q(t)$, which is the solution of the equation of motion, provides an extremum of the action [under the condition (2.5)]. Note that with the help of the Lagrange function the energy and momentum of the particle can also be obtained:

$$H = \frac{m\dot{q}^2}{2} + V = \frac{\partial L}{\partial \dot{q}}\dot{q} - L, \qquad p = m\dot{q} = \frac{\partial L}{\partial \dot{q}}. \tag{2.8}$$

We now turn to an examination of classical fields. A well-known example is the electromagnetic field which is described by the potential $A_\mu(\boldsymbol{x}, t)$ ($\boldsymbol{A}$ is the vector potential, $A_0 = -iA_4$ is the scalar potential). This means that the electromagnetic field is described by an infinite number of functions of time (the values of the potential at all points of space), i.e. the electromagnetic field is a system with an infinite number of degrees of freedom. The equations of the field (Maxwell's equations) can be obtained from the variational principle which is a general principle applicable to any physical system.

We shall examine fields in addition to the electromagnetic field. Denote the functions describing some field by $\psi_\alpha(x)$, where the index α assumes integer values and $x = (\boldsymbol{x}, ix_0)$, $x_0 \equiv t$. We write the Lagrangian of the field in the form

$$L = \int \mathcal{L}\left(\psi, \frac{\partial \psi}{\partial x}\right) d\boldsymbol{x}. \tag{2.9}$$

Here $\mathcal{L}(\psi, \partial\psi/\partial x)$ is the Lagrangian of a unit volume (the Lagrangian density).† We assumed that this quantity depends only on the functions $\psi_\alpha(x)$ and their first derivatives $\partial\psi_\alpha/\partial x_\mu$ [compare (2.3)]. Define the action

$$\mathcal{S} = \int_\Omega \mathcal{L}\left(\psi, \frac{\partial \psi}{\partial x}\right) dx, \tag{2.10}$$

where $dx = d\boldsymbol{x}\, dx_0$ and Ω is some volume of space-time. The variation of the action is

† Henceforth we shall call this quantity the Lagrangian.