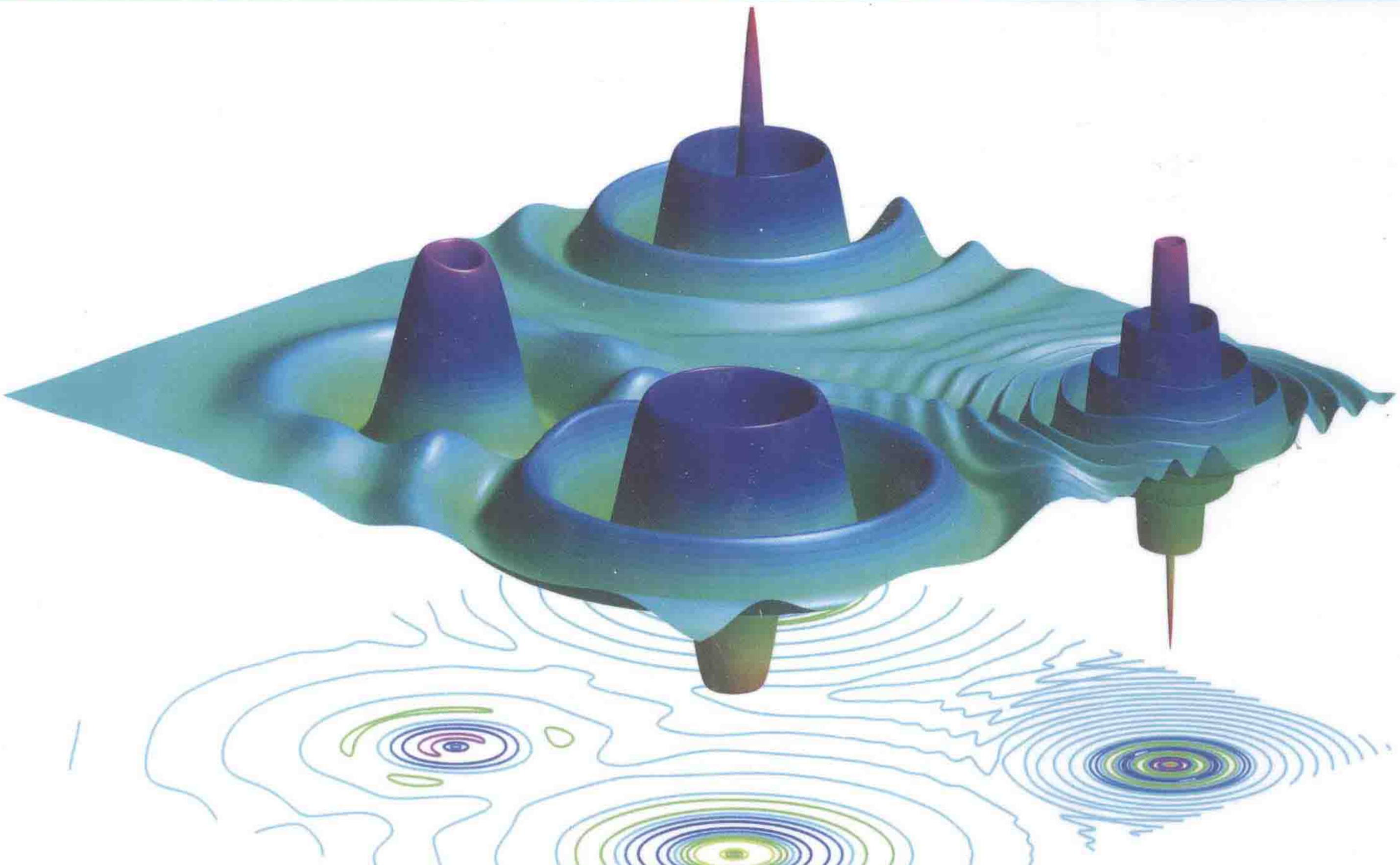
Hydrodynamics of Time-Periodic Groundwater Flow

Diffusion Waves in Porous Media



Joe S. Depner and Todd C. Rasmussen

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PREFACE

Goal and Purpose

Our goal in writing this book is to present a clear and accessible mathematical introduction to the basic theory of time-periodic groundwater flow. Understanding the basic theory is essential for those who seek a comprehensive knowledge of groundwater hydraulics and groundwater hydrology. In addition, the basic theory has an aesthetic beauty that readers can learn to appreciate and thereby enjoy.

Intended Audience

We intend this book to be used primarily for self-directed study by advanced undergraduate and graduate students and by working scientists and engineers in the earth and environmental sciences. This book is suitable for well-prepared readers who either (a) are new to the field of periodic groundwater flow and seek a formal introduction to the theory or (b) were introduced to the field in the distant past and wish to renew their knowledge and enrich their understanding. Additionally, we hope that this book will be a useful resource for educators.

The mathematical framework for time-periodic groundwater flow is structurally equivalent to that of time-periodic diffusion. Therefore, some of the theory presented in this book may be relevant to time-periodic phenomena encountered in fields other than groundwater flow, like electrical conduction, thermal conduction, and molecular diffusion. Consequently, we expect that students and professionals in these other fields also will find parts of this book useful.

Prerequisites

We assume that the reader has completed university courses in multivariable calculus, linear algebra, and subsurface fluid dynamics (e.g., groundwater hydraulics). Also, the reader should have a *basic* familiarity with complex variables, Fourier series, and partial differential equations (PDEs). Readers do not need to know contour integration in the complex plane or Green functions.

Approach

Our development is quantitative. We emphasize problem definition and problem understanding, rather than problem solution techniques, because we believe the former are fundamental prerequisites of the latter and because solution techniques have been described exhaustively by other authors (e.g., *Carslaw and Jaeger* [1986], *Özişik* [1989], *Hermance* [1998], *Bruggeman* [1999], *Mandelis* [2001]).

Much of the information presented here could be gleaned from reading articles in peer-reviewed scientific publications such as those listed in the Bibliography. However, one would have to read many such articles, which typically present only terse descriptions of the mathematical development. This book is more explicit to accommodate the needs of those who are new to the field of periodic groundwater flow. It shows more intermediate steps so that readers can follow the logic of the development, understand the mathematical context, and recognize the limitations of the approach.

Scope

Assumptions

The scope of this book is limited to time-periodic flows of homogeneous fluids through fully saturated, elastically deformable, porous media in which Darcy's law is satisfied. Within this scope, we have attempted to present the basic theory in a general form so that the results are widely applicable. To that end we make the following basic assumptions, among others:

- The relevant space domain is N-dimensional, where N can be 1, 2, or 3.
- The porous medium is macroscopically nonhomogeneous (i.e., spatially nonuniform) with respect to material hydrologic properties. That is, the medium's hydraulic conductivity and specific storage are functions of the space coordinates.
- In multidimensional cases, the porous medium generally is nonuniformly anisotropic with respect to hydraulic conductivity. That is, the problem under consideration cannot be transformed to one in which the anisotropic medium is replaced by an equivalent, macroscopically isotropic one simply by linearly transforming the space coordinates.
- The periodic component of the forcing need not be strictly periodic; it may be *almost periodic* (see Section 1.1 for a discussion of relevant terminology).

With the exception of some illustrative examples and exercises, which we clearly identify, we adhere to these assumptions throughout this book.

Organization

This book consists of the following parts:

- Part I (Introduction, Chapter 1) introduces basic terminology, proposes criteria for the classification of time-periodic forcing, and lists potential areas of application for the theory of time-periodic groundwater flow.
- Part II (Problem Definition, Chapters 2–8) describes the conceptual, mathematical basis of periodic groundwater flow within the framework of the classical boundary-value problem (BVP). It lays the foundation for subsequent parts.
- Part III (Elementary Examples, Chapters 9–13) presents examples of elementary solutions of the complex-variable form of the space BVP.
- Part IV (Essential Concepts, Chapters 14 and 15) explores some basic concepts of periodic flow, such as attenuation, delay, and local time variation of the specific discharge.
- Part V (Stationary Points, Chapters 16–18) examines the existence and nature of stationary points of the hydraulic head amplitude and phase functions and their relation to flow stagnation.
- Part VI (Wave Propagation, Chapters 19–22) presents a conceptualization of periodic groundwater flow as propagation of spatially attenuated (damped), traveling diffusion waves, i.e., harmonic, hydraulic head waves.
- Part VII (Energy Transport, Chapters 23–25) explores the transport of fluid mechanical energy by periodic groundwater flow under isothermal conditions.
- Part VIII (Conclusion, Chapter 26) briefly summarizes the results obtained in the preceding chapters, unresolved issues, and limitations of the book.

Suggested Use

The chapters are meant to be read sequentially within each part.

We believe that all readers should begin by studying Parts I and II. This material forms the core of the subject and is prerequisite for learning about the more advanced topics presented in subsequent parts. After studying Part II, readers should at least browse Parts III and IV to familiarize themselves with their scope. The reader's subsequent course of action depends on individual preference. Those who have both the interest and sufficient time should read all of the remaining parts sequentially. Those who are pressed for time or whose interests are more limited may study a combination of Parts V–VII. Lastly, all readers will want to read Part VIII.

We have embedded more than 360 exercises (see List of Exercises) in the text and included the solutions for nearly all. Each exercise is numbered and accompanied by a title that briefly summarizes its topic. The exercises are intended to reinforce the ideas presented in the text and in many cases are essential elements of the theoretical development. Exercises typically emphasize abstract reasoning, requiring symbolic manipulation rather than numerical computation. Believing that most readers will be more familiar with the material in the earlier parts than that in the later parts, we have placed more exercises in the later parts. Ideally, less advanced readers should attempt to complete every exercise they encounter. More advanced readers and those pressed for time should, at minimum, carefully read each exercise and the accompanying solution to maintain the flow of the presentation.

Usability

The electronic version of this book employs the following features for reader convenience:

- All book components (parts, chapters, sections, subsections, appendices) listed in the Contents are digitally "bookmarked." This allows the reader to navigate to the beginning of any such component, from any point in the book, via hyperlink. To activate a bookmark hyperlink, click on the corresponding label in the bookmark's navigation panel.
- In-line hyperlinks are used extensively. These include references to the following items:
  - Worldwide website uniform resource locators (URLs)
  - Specific book components: parts, chapters, sections, subsections, appendices, etc.
  - Specific content features: equations, examples, exercises, figures, notes, tables
  - Page references in the keyword index

Hyperlinked references, both in the table of contents and elsewhere, appear as blue-colored text. To activate an in-line hyperlink, click on the corresponding blue-colored text.

- Important terms appear in *italics* to draw the reader's attention.

• Examples, exercises, and notes appear with translucent shaded backgrounds colored blue violet, yellow, and green, respectively, to help the reader quickly recognize them.

Website and Contacts

Readers are invited to help improve the quality of this book by reporting errors and suggesting changes. For details, visit the companion website at hydrology.uga.edu/periodic/.

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NOTATION

Latin Symbols

Symbol	Description
A	Coefficient of cosine term in Fourier series; real component of complex amplitude; dimensionless coefficient in frequency response function for one-dimensional flow
$\mathbf{A}$	Coefficient matrix of linear differential operator
adj	Adjugate matrix
Arctan	Principal value of inverse-tangent function
arctan	Arctangent (inverse-tangent) function
arg	Argument of complex number
B	Coefficient of sine term in Fourier series; imaginary component of complex amplitude; dimensionless coefficient in frequency response function for one-dimensional flow
$\mathbf{b}$	Coefficient vector of linear differential operator
BC	Boundary condition
ber_ν, bei_ν	Kelvin functions, b-type, order ν ($\nu \in \mathbb{R}$)
BVP	Boundary value problem
$\mathbf{C}$	Matrix of linear transformation of space coordinates
c	Constant; propagation speed of harmonic, traveling wave
$\mathbf{c}$	Eigenvector of FRF for uniform-gradient flow in exponential media
c_h	Coefficient of hydraulic head term in boundary condition equation; propagation speed of hydraulic head constituent wave
$c^{(n)}$	Propagation speed of nth-component wave (n integer, $n > 0$)
c_q	Coefficient of specific-discharge term in boundary condition equation
const	Constant
cos	Cosine function
cosh	Hyperbolic cosine function
1D	One dimensional
2D	Two dimensional
3D	Three dimensional
d	Ordinary differentiation operator
dB	Decibel(s)
det	Determinant of matrix
D	N-dimensional space domain
$D_0(\omega_m)$	Zero set of hydraulic head constituent amplitude $M_\mathrm{h}(\mathbf{x}; \omega_m)$
$D_+(\omega_m)$	Cozero set of hydraulic head constituent amplitude $M_\mathrm{h}(\mathbf{x}; \omega_m)$
E	Fluid mechanical energy
e	Void ratio
e	Euler's number (also, Napier's constant), the mathematical constant, $e = 2.7182818\ldots$
$\mathbf{e}_m$	Unit basis vector for Cartesian coordinate x_m ($m = 1, 2, 3$)
exp	Exponentiation operator, i.e., $\exp(\,) = e^{(\,)}$
F	Complex amplitude
$F^{(n)}$	Frequency response eigenfunction
FRE	Frequency response eigenfunction
FRF	Frequency response function
g	Acceleration of gravity
G	Space domain
$G(r)$, $G(x)$	Frequency response function

Symbol	Description
$\cap$	Intersection of sets
$\cup$	Union of sets
$\varnothing$	Empty set
$\forall$	For all
:	Such that
:	Matrix double inner product
|	Such that
| |	Absolute value of real number; modulus of complex number; magnitude (L^2-norm) of n-dimensional vector
$\cdot$	Inner (scalar) product on finite-dimensional vector space
$\times$	Cross (vector) product
$\rightarrow$	Approaches
$\Rightarrow$	Implies
$\langle\ \rangle$	Inner product on infinite-dimensional vector space; time average
$\{\ \}$	Set

Symbol	Description
N	Dimension of space domain ($N = 1, 2, 3$)
$N(\omega_m)$	Number of component waves for the mth-harmonic constituent (N integer, $N > 0$)
N_ν	Kelvin modulus function, k-type, order ν
$\hat{\mathbf{n}}$	Unit vector outwardly perpendicular to space domain boundary
$(\ )^{(n)}$	Component wave index (n integer, $n > 0$)
O	Order of
ODE	Ordinary differential equation
p	Fluid pressure
$p_m^{(n)}$	Dimensionless exponent of FRF for power law media (m, n integer; $m, n > 0$)
PDE	Partial differential equation
$\mathbf{Q}$	Complex-valued, harmonic constituent (vector) of periodic transient component of groundwater specific discharge
Q	Quality factor
Q_0	Complex amplitude of point source
ΔQ_0	Complex amplitude of line source or plane source
$\mathbf{q}$	Specific discharge (vector)
$\mathbf{q}'$	Harmonic constituent (vector) of periodic transient component of groundwater specific discharge
$(\ )_{\mathrm{Qu}}$	Quadrature component of harmonic constituent
$\mathbb{R}$	Set of all real numbers
R	Dimensionless radial space coordinate
rad	Radian
r_{D}	Dimensionless envelope of specific-discharge constituent
Re	Real part of complex number
RHS	Right-hand side (of equation or inequality)
S	Control surface
s	Second
s	Wave travel distance
$s^{(n)}$	Travel distance for component wave (n integer, $n > 0$)
$\hat{\mathbf{s}}_{\mathrm{h}}$	Unit ray path vector of hydraulic head constituent wave
$\hat{\mathbf{s}}^{(n)}$	Unit ray path vector of nth-component wave (n integer, $n > 0$)
sech	Hyperbolic secant function
sign	Sign function
sin	Sine function
sinh	Hyperbolic sine function
S_{s}	Specific storage of porous medium
T	Travel time; fluid temperature
t	Time (dimension or independent variable)
T_m	Period of mth-harmonic constituent (m integer, $m > 0$)
$T^{(n)}$	Travel time of nth-component wave (n integer, $n > 0$)
tr	Trace of matrix
$(\ )^{\mathrm{T}}$	Matrix transpose
U	Complex-valued harmonic constituent of periodic transient component of groundwater volumetric source strength
u'	Harmonic constituent of periodic transient component of groundwater volumetric source strength
$\hat{\mathbf{u}}$	Vector of unit length pointing in the direction of $\mathbf{K}\nabla M_{\mathrm{h}}$
URL	Uniform resource locator
V	Control volume
$\mathbf{v}$	Nominal seepage velocity; phase velocity of traveling wave
$\mathbf{v}_{\mathrm{h}}$	Phase velocity of hydraulic head constituent wave
$\hat{\mathbf{v}}$	Vector of unit length pointing in the direction of $\mathbf{K}\nabla\theta_{\mathrm{h}}$

Symbol	Description
G_D	Dimensionless frequency response function
$G^{(n)}$	Frequency response eigenfunction
$\mathbf{H}[f(\mathbf{x})]$	Hessian matrix of the function $f(\mathbf{x})$
H	Complex-valued harmonic constituent of periodic transient component of hydraulic head
h	Hydraulic head
h'	Harmonic constituent of periodic transient component of hydraulic head
$\mathbf{I}$	Identity tensor
I_0	Modified Bessel function of the first kind, order zero
IBVP	Initial boundary value problem
IC	Initial condition
Im	Imaginary part of complex number
i	Imaginary unit (i.e., $i = \sqrt{-1}$)
i_0	Modified spherical Bessel function of the first kind, order zero
$(\ )_{\mathrm{Ip}}$	In-phase component of harmonic constituent
$\mathbf{J}$	Fluid mechanical energy flux density (vector)
$\mathbf{K}$	Hydraulic conductivity (tensor) of porous medium
K	Hydraulic conductivity (scalar) of porous medium
K_0	Modified Bessel function of the second kind, order zero
$K_{-1/2}$	Modified Bessel function of the second kind, order $-1/2$
k	Wave number
$k^{(n)}$	Wave number for nth-component wave (n integer, $n > 0$)
$\mathbf{k}$	Wave vector
$\mathbf{k}^{(n)}$	Wave vector for nth-component wave (n integer, $n > 0$)
k_0	Modified spherical Bessel function of the second kind, order zero
ker_ν, kei_ν	Kelvin functions, k-type, order ν ($\nu \in \mathbb{R}$)
kg	Kilogram
L	Length (dimension)
$L^{(n)}$	Penetration depth for nth-component wave (n integer, $n > 0$)
$L[\ ]$	Linear differential operator
$L_h[\ ]$	Linear differential operator
$L_M[\ ]$	Linear differential operator
$L_\theta[\ ]$	Linear differential operator
LHS	Left-hand side (of equation or inequality)
l	Length of one-dimensional space domain
lim	Limit
ln	Natural (base-e) logarithm
$\log_{10}$	Base-10 logarithm
M	Mass (dimension)
M_h	Amplitude function for hydraulic head harmonic constituent
$M^{(n)}$	Amplitude function for nth-component wave (n integer, $n > 0$)
M_u	Amplitude function for source term harmonic constituent
M_ψ	Amplitude function for boundary condition harmonic constituent
M_ν	Kelvin modulus function, b-type, order ν ($\nu \in \mathbb{R}$)
m	Harmonic constituent index (m integer, $m > 0$)
m	Meter
max	Maximum value
$\max_t$	Maximum value with respect to time
ME	Mechanical energy
min	Minimum value
N	Newton

Symbol	Description
$\boldsymbol{\mu}$	Gradient (vector) of natural logarithm of hydraulic conductivity or specific storage
ν	Dimensionless length scale of hydraulic conductivity or specific storage; order of Bessel function or Kelvin function
ξ	Wave phase
ξ_{h}	Wave phase for hydraulic head constituent wave
$\xi^{(n)}$	Wave phase for nth-component wave (n integer, $n > 0$)
π	The mathematical constant, $\pi = 3.14159265\ldots$
ρ_{s}	Volumetric mass density of solid phase of porous medium
ρ_{w}	Volumetric mass density of groundwater
Σ	Summation
σ	Local, per-volume rate of delivery of fluid mechanical energy by internal source(s)
σ_{e}	Effective stress
ϕ	Initial-value function for hydraulic head; porosity
ϕ_{ν}	Kelvin phase function, k-type, order ν ($\nu \in \mathbb{R}$)
ϕ_{e}	Effective porosity
Ψ	Complex-valued harmonic constituent of periodic transient component of boundary value function
ψ	Boundary value function
ψ'	Harmonic constituent of periodic transient component of boundary value function
ω	Angular frequency
ω_m	Angular frequency of mth-harmonic constituent (m integer, $m > 0$)

Other Symbols

Symbol	Description
$\bar{}$	(bar accent) Steady component
$\tilde{}$	(tilde accent) Transient component
$\hat{}$	(hat accent) Nonperiodic transient component
$\mathring{}$	(ring accent) Time-periodic transient component
$\breve{}$	(breve accent) Transformed variable
$*$	(asterisk) Complex conjugate
$\mathbf{0}$	Zero vector
$(\ )^{-1}$	Reciprocal; matrix inverse
$\int dv$	Integration with respect to the variable v
∞	Infinity
∂	Partial differentiation operator
∇	N-dimensional spatial gradient operator (vector)
$\nabla\cdot$	N-dimensional spatial divergence operator (scalar)
∇^2	N-dimensional Laplacian operator (scalar)
∇^4	Fourth-order, N-dimensional Laplacian operator (scalar)
$\sqrt{}$	Positive square root
$=$	Is equal to
$\approx$	Is approximately equal to
$<$	Is less than
$\ll$	Is much less than
$>$	Is greater than
$\gg$	Is much greater than
$\equiv$	Is defined as
$\in$	Is an element of the set
$\subset$	Is a subset of the set

Symbol	Description
$\mathbf{v}^{(n)}$	Phase velocity of nth-component wave (n integer, $n > 0$)
$W[\ ]$	Linear differential operator for boundary condition equation
X	Dimensionless space coordinate
$\mathbf{x}$	N-dimensional space coordinate vector
$\hat{\mathbf{x}}$	Unit basis vector for Cartesian space coordinate x (also, $\mathbf{e}_1$)
x	Cartesian space coordinate (also, x_1)
x_m	Cartesian space coordinate ($m = 1, 2, 3$)
Y_h	Logarithm of amplitude of hydraulic head harmonic constituent
$\mathbb{Z}$	Set of all integers
z	Complex variable

Greek Symbols

Symbol	Description
α	Bulk compressibility of porous medium
α_m	Constituent parameter [see equation (5.31)] (m integer, $m > 0$)
β	Coefficient of order-zero term in generalized wave equation
β_w	Isothermal compressibility of liquid water
Γ	Boundary of space domain
Γ_h	Subset of the boundary for which the pertinent boundary condition is of the Dirichlet type
Γ_{hq}	Subset of the boundary for which the pertinent boundary condition is of the Robin type
$\Gamma_m^{(n)}$	Dimensionless eigenvalue (m, n integer; m, $n > 0$)
$\Gamma^{(n)}$	Ray path geometric divergence for nth-component wave (n integer, $n > 0$)
Γ_q	Subset of the boundary for which the pertinent boundary condition is of the Neumann type
$\Gamma_+(\omega_m)$	Boundary of subregion $D_+(\omega_m)$
γ	Coefficient of order-one term in generalized wave equation
δ_h	Spatial attenuation scale for hydraulic head harmonic constituent
δ_{mn}	Kronecker delta (sometimes referred to as *Kronecker's delta*) (m, n integer; m, $n > 0$)
$\delta^{(n)}$	Spatial attenuation scale for nth-component wave (n integer, $n > 0$)
ϵ	Fluid mechanical energy density
ϵ_{ijk}	Three-dimensional Levi-Civita symbol
ζ_h	Space derivative of hydraulic head harmonic constituent phase function in one-dimensional flow
$\zeta^{(n)}$	Space derivative of nth-component wave phase function in one-dimensional flow (n integer, $n > 0$)
η	Local, per-volume, fluid mechanical energy dissipation rate
η_m	Dimensionless parameter used for plotting dimensionless FRF solutions (m integer, $m > 0$)
θ_D	Phase function for dimensionless hydraulic head frequency response function
θ_h	Phase function for hydraulic head harmonic constituent
$\theta^{(n)}$	Phase function for nth-component wave (n integer, $n > 0$)
θ_u	Phase function for source term harmonic constituent
θ_ψ	Phase function for boundary condition harmonic constituent
θ_ν	Kelvin phase function, b-type, order ν ($\nu \in \mathbb{R}$)
κ	Hydraulic diffusivity
Λ	Logarithm of bulk compressibility
$\Lambda_m^{(1)}, \Lambda_m^{(2)}$	Dimensionless eigenvalues for mth constituent (m integer, $m > 0$)
λ	Wavelength
$\lambda_m, \lambda_m^{(1)}, \lambda_m^{(2)}$	Eigenvalues for mth constituent (m integer, $m > 0$)
$\lambda^{(n)}$	Local wavelength of nth-component wave (n integer, $n > 0$)
μ	Reciprocal length scale for spatial variation of natural logarithm of hydraulic conductivity or specific storage

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CONTENTS

Part I
Introduction