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Complex Manifolds and Deformation of Complex Structures

Translated by Kazuo Akao

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Preface

This book is an introduction to the theory of complex manifolds and their deformations.

Deformation of the complex structure of Riemann surfaces is an idea which goes back to Riemann who, in his famous memoir on Abelian functions published in 1857, calculated the number of effective parameters on which the deformation depends. Since the publication of Riemann's memoir, questions concerning the deformation of the complex structure of Riemann surfaces have never lost their interest.

The deformation of algebraic surfaces seems to have been considered first by Max Noether in 1888 (M. Noether: *Anzahl der Modulen einer Classe algebraischer Flächen*, Sitz. Königlich. Preuss. Akad. der Wiss. zu Berlin, erster Halbband, 1888, pp. 123–127). However, the deformation of higher dimensional complex manifolds had been curiously neglected for 100 years. In 1957, exactly 100 years after Riemann's memoir, Frölicher and Nijenhuis published a paper in which they studied deformation of higher dimensional complex manifolds by a differential geometric method and obtained an important result. (A. Frölicher and A. Nijenhuis: A theorem on stability of complex structures, *Proc. Nat. Acad. Sci., U.S.A.*, **43** (1957), 239–241).

Inspired by their result, D. C. Spencer and I conceived a theory of deformation of compact complex manifolds which is based on the primitive idea that, since a compact complex manifold M is composed of a finite number of coordinate neighbourhoods patched together, its deformation would be a shift in the patches. Quite naturally it follows from this idea that an infinitesimal deformation of M should be represented by an element of the cohomology group $H^1(M, \Theta)$ of M with coefficients in the sheaf Θ of germs of holomorphic vector fields. However, there seemed to be no reason that any given element of $H^1(M, \Theta)$ represents an infinitesimal deformation of M. In spite of this, examination of familiar examples of compact complex manifolds M revealed a mysterious phenomenon that $\dim H^1(M, \Theta)$ coincides with the number of effective parameters involved in the definition of M. In order to clarify this mystery, Spencer and I developed the theory of deformation of compact complex manifolds. The process of the development was the most interesting experience in my whole mathematical life. It was similar to an experimental science developed by the interaction between experiments (examination of examples) and theory. In this book I have tried to reproduce this interesting experience; however I could not fully convey it. Such an experience may be a passing phenomenon which cannot be reproduced.

The theory of deformation of compact complex manifolds is based on the theory of elliptic partial differential operators expounded in the Appendix. I would like to express my deep appreciation to Professor D. Fujiwara who kindly wrote the Appendix and also to Professor K. Akao who spent the time and effort translating this book into English.

Tokyo, Japan
January, 1985

KUNIHIKO KODAIRA

Contents

Chapter 1

Holomorphic Functions

§1.1. Holomorphic Functions

(a) Holomorphic Functions

We begin by defining holomorphic functions of n complex variables. The n-dimensional complex number space is the set of all n-tuples $(z_1, \ldots, z_n)$ of complex numbers z_i, $i = 1, \ldots, n$, denoted by $\mathbb{C}^n$. $\mathbb{C}^n$ is the Cartesian product of n copies of the complex plane: $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$. Denoting $(z_1, \ldots, z_n)$ by z, we call $z = (z_1, \ldots, z_n)$ a point of $\mathbb{C}^n$, and $z_1, \ldots, z_n$ the complex coordinates of z. Letting $z_j = x_{2j-1} + ix_{2j}$ by decomposing z_j into its real and imaginary parts (where $i = \sqrt{-1}$), we can express z as

$$z = (x_1, x_2, \ldots, x_{2n-1}, x_{2n}). \tag{1.1}$$

Thus $\mathbb{C}^n$ is considered as the $2n$-dimensional real Euclidean space $\mathbb{R}^{2n}$ equipped with the complex coordinates. $x_1, x_2, \ldots, x_{2n-1}, x_{2n}$ are called the real coordinates of z. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$. We define the linear combination $\lambda z + \mu w$ of z and w, viewed as vectors, by

$$\lambda z + \mu w = (\lambda z_1 + \mu w_1, \ldots, \lambda z_n + \mu w_n),$$

where λ and μ are complex numbers. This makes $\mathbb{C}^n$ a complex linear space. The length of $z = (z_1, \ldots, z_n)$ is defined by

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}. \tag{1.2}$$

Clearly we have

$$|\lambda z| = |\lambda||z|, \tag{1.3}$$

$$|z + w| \leqq |z| + |w|. \tag{1.4}$$

The distance of the two points $z, w \in \mathbb{C}^n$ is given by

$$|z - w| = \sqrt{|z_1 - w_1|^2 + \cdots + |z_n - w_n|^2}. \tag{1.5}$$

We introduce a topology on $\mathbb{C}^n$ by the identification with $\mathbb{R}^{2n}$ with the usual topology. Thus, for example, a subset $D \subset \mathbb{C}^n$ is a domain in $\mathbb{C}^n$ if D is a domain considered as a subset of $\mathbb{R}^{2n}$. Again, a complex-valued function $f(z) = f(z_1, \ldots, z_n)$ defined on a subset D in $\mathbb{C}^n$ is continuous if $f(z)$ is so as a function of the real coordinates $x_1, x_2, \ldots, x_{2n}$.

Now we consider a complex-valued function $f(z) = f(z_1, \ldots, z_n)$ of n complex variables $z_1, \ldots, z_n$ defined on a domain $D \subset \mathbb{C}^n$.

Definition 1.1. If $f(z) = f(z_1, \ldots, z_n)$ is continuous in $D \subset \mathbb{C}^n$, and holomorphic in each variable z_k, $k = 1, \ldots, n$, separately, $f(z_1, \ldots, z_n)$ is said to be *holomorphic* in D. We also call $f(z) = f(z_1, \ldots, z_n)$ a *holomorphic function of n variables* $z_1, \ldots, z_n$.

Here, by saying that $f(z_1, \ldots, z_k, \ldots, z_n)$ is holomorphic in z_k separately, we mean that $f(z_1, \ldots, z_n)$ is a holomorphic function in z_k when the other variables $z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n$ are fixed.

The fundamental Cauchy integral formula with respect to a circle for holomorphic functions of one variable is extended to the case of holomorphic functions of n variables as follows.

Given a point $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ and positive real numbers $r_1, \ldots, r_n$, we put

$$U_r(c) = \{z \mid z = (z_1, \ldots, z_n) \mid |z_k - c_k| < r_k, k = 1, \ldots, n\}, \tag{1.6}$$

where r denotes $(r_1, \ldots, r_n)$. Let $U_{r_k}(c_k)$ be the disk with centre c_k and radius r_k on the z_k-plane. Then we have

$$U_r(c) = U_{r_1}(c_1) \times \cdots \times U_{r_n}(c_n). \tag{1.7}$$

Thus we call $U_r(c)$ the polydisk with centre c. We denote by C_k the boundary of $U_{r_k}(c_k)$, that is, the circle of radius r_k with centre c_k on the z_k-plane. Of course C_k is represented by the usual parametrization $\theta_k \to \gamma(\theta_k) = c_k + r_k e^{i\theta_k}$ where $0 \leqq \theta_k \leqq 2\pi$. The product of $C_1, C_2, \ldots, C_n$

$$C^n = C_1 \times \cdots \times C_n \tag{1.8}$$

is called the determining set of the polydisk $U_r(c)$. C^n is an n-dimensional torus. Given a continuous function $\psi(\zeta) = \psi(\zeta_1, \ldots, \zeta_n)$, with $\zeta_1 \in C_1, \ldots, \zeta_n \in C_n$, we define its integral over C^n by

$$\int_{C^n} \psi(\zeta)\, d\zeta_1 \cdots d\zeta_n = \int_{C_1} \cdots \int_{C_n} \psi(\zeta)\, d\zeta_1 \cdots d\zeta_n$$

$$= \int_0^{2\pi} \cdots \int_0^{2\pi} \psi(\gamma_1(\theta_1), \ldots, \gamma_n(\theta_n))\gamma_1'(\theta_1) \ldots \gamma_n'(\theta_n)\, d\theta_1 \ldots d\theta_n. \tag{1.9}$$

Theorem 1.1. *Let $f=f(z_1,\dots,z_n)$ be a holomorphic function in a domain $D\subsetneq\mathbb{C}^n$. Take a polydisk $U_r(c)$ with $[U_r(c)]\subset D$. Then for $z\in U_r(c)$, $f(z)$ is represented as*

$$f(z)=\left(\frac{1}{2\pi i}\right)^n\int_{C^n}\frac{f(\zeta_1,\dots,\zeta_n)}{(\zeta_1-z_1)\cdots(\zeta_n-z_n)}\,d\zeta_1\cdots d\zeta_n,\qquad(1.10)$$

where $[\cdot]$ *denotes the closure.*

Proof. First we consider the case $n=2$. In this case the right-hand side of (1.10) becomes

$$\left(\frac{1}{2\pi i}\right)^2\int_0^{2\pi}\int_0^{2\pi}\frac{f(\gamma_1(\theta_1),\gamma_2(\theta_2))\gamma_1'(\theta_1)\gamma_2'(\theta_2)}{(\gamma_1(\theta_1)-z_1)(\gamma_2(\theta_2)-z_2)}\,d\theta_1\,d\theta_2,$$

where the integrand is a continuous function of θ_1 and θ_2 for $(z_1,z_2)\in U_r(c)$. Hence by the formula of the iterated integral, this integral is equal to

$$\left(\frac{1}{2\pi i}\right)^2\int_0^{2\pi}\frac{\gamma_1'(\theta_1)}{\gamma_1(\theta_1)-z_1}\,d\theta_1\int_0^{2\pi}\frac{f(\gamma_1(\theta_1),\gamma_2(\theta_2))\gamma_2'(\theta_2)}{\gamma_2(\theta_2)-z_2}\,d\theta_2.$$

Therefore by the Cauchy integral formula, the right-hand side of (1.10) becomes

$$\left(\frac{1}{2\pi i}\right)^2\int_{C_1}\frac{1}{\zeta_1-z_1}\,d\zeta_1\int_{C_2}\frac{f(\zeta_1,\zeta_2)}{\zeta_2-z_2}\,d\zeta_2$$
$$=\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta_1,z_2)}{\zeta_1-z_1}\,d\zeta_1=f(z_1,z_2),$$

which proves (1.10) in this case. Similarly for general n, by a repeated application of the Cauchy integral formula, the right-hand side of (1.10) becomes

$$\left(\frac{1}{2\pi i}\right)^n\int_{C_1}\frac{d\zeta_1}{\zeta_1-z_1}\cdots\int_{C_{n-1}}\frac{d\zeta_{n-1}}{\zeta_{n-1}-z_{n-1}}\int_{C_n}\frac{f(\zeta_1,\dots,\zeta_{n-1},\zeta_n)}{\zeta_n-z_n}\,d\zeta_n$$
$$=\left(\frac{1}{2\pi i}\right)^{n-1}\int_{C_1}\frac{d\zeta_1}{\zeta_1-z_1}\cdots\int_{C_{n-1}}\frac{f(\zeta_1,\dots,\zeta_{n-1},z_n)}{\zeta_{n-1}-z_{n-1}}\,d\zeta_{n-1}$$
$$=\cdots=\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta_1,z_2,\dots,z_n)}{\zeta_1-z_1}\,d\zeta_1=f(z_1,\dots,z_n).\quad\blacksquare$$

As in the case of holomorphic functions of one variable, we shall deduce the fundamental properties of holomorphic functions of n variables from the integral formula (1.10).

First let $\psi(\zeta_1, \ldots, \zeta_n)$ be a continuous function on $C^n = C_1 \times \cdots \times C_n$, and $m_1, \ldots, m_n$ natural numbers. Consider the integral

$$g(z) = \int_{C^n} \frac{\psi(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{m_1} \cdots (\zeta_n - z_n)^{m_n}} \tag{1.11}$$

as a function of $z = (z_1, \ldots, z_n) \in U_r(c)$. Clearly $g(z)$ is continuous in $U_r(c)$. Then for fixed $z_2 \in U_{r_2}(c_1), \ldots, z_n \in U_{r_n}(c_n)$, put

$$\varphi(\zeta_1) = \int_{C_2} \cdots \int_{C_n} \frac{\psi(\zeta_1, \ldots, \zeta_n)\, d\zeta_2 \cdots d\zeta_n}{(\zeta_2 - z_2)^{m_2} \cdots (\zeta_n - z_n)^{m_n}}.$$

$\varphi(\zeta_1)$ is a continuous function of ζ_1 on C_1. Hence

$$g(z) = g(z_1, z_2, \ldots, z_n) = \int_{C_1} \frac{\varphi(\zeta_1)}{(\zeta_1 - z_1)^{m_1}}\, d\zeta_1 \tag{1.12}$$

is a holomorphic function of z_1 in $U_{r_1}(c_1)$. Similarly $g(z_1, \ldots, z_n)$ is a holomorphic function of each variable z_k, $k = 1, \ldots, n$, in $U_{r_k}(c_k)$. Hence $g(z) = g(z_1, \ldots, z_n)$ is a holomorphic function of n variables $z_1, \ldots, z_n$ in the polydisk $U_r(c)$. By (1.12) we have

$$\begin{aligned}\frac{\partial}{\partial z_1} g(z_1, \ldots, z_n) &= m_1 \int_{C_1} \frac{\varphi(\zeta_1)}{(\zeta_1 - z_1)^{m_1+1}}\, d\zeta_1 \\ &= m_1 \int_{C_1} \cdots \int_{C_n} \frac{\psi(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{m_1+1} \cdots (\zeta_n - z_n)^{m_n}}.\end{aligned} \tag{1.13}$$

Thus $(\partial g/\partial z_1)(z_1, \ldots, z_n)$ is also holomorphic in $U_r(c)$. Similar results hold also for $\partial g/\partial z_k$.

By a repeated application of this result to the right-hand side of (1.10), we obtain the following theorem.

Theorem 1.2. *A holomorphic function $f(z) = f(z_1, \ldots, z_n)$ of n variables in a domain $D \subset \mathbb{C}^n$ is arbitrarily many times differentiable in $z_1, \ldots, z_k$ in D, and all its partial derivatives $\partial^{m_1+\cdots+m_n} f(z)/\partial z_1^{m_1} \cdots \partial z_n^{m_n}$ are holomorphic in D. Moreover taking a polydisk $U_r(c)$ such that $[U_r(c)] \subset D$, we have*

$$\begin{aligned}&\frac{\partial^{m_1+\cdots+m_n}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} f(z_1, \ldots, z_n) \\ &\qquad = \frac{m_1! \cdots m_n!}{(2\pi i)^n} \int_{C^n} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1)^{m_1+1} \cdots (\zeta_n - z_n)^{m_n+1}}\end{aligned} \tag{1.14}$$

in $U_r(c)$. ∎

As in the case of functions of one variable we denote by $f^{(m_1 \cdots m_n)}(z_1, \ldots, z_n)$ the partial derivative

$$\frac{\partial^{m_1+\cdots+m_n}}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} f(z_1, \ldots, z_n) \quad \text{of} \quad f(z) = f(z_1, \ldots, z_n).$$

Theorem 1.3. *Let $f(z) = f(z_1, \ldots, z_n)$ be a holomorphic function in a domain $D \subset \mathbb{C}^n$, and $c = (c_1, \ldots, c_n) \in D$. Then in a polydisk $U_{\rho(c)} \subset D$ with centre c, $f(z)$ has a power series expansion in $z_1 - c_1, \ldots, z_n - c_n$,*

$$f(z) = \sum_{m_1, \ldots, m_n = 0}^{\infty} a_{m_1 \cdots m_n} (z_1 - c_1)^{m_1} \cdots (z_n - c_n)^{m_n}, \tag{1.15}$$

which is absolutely convergent in $U_\rho(c)$. The coefficient $a_{m_1 \cdots m_n}$ is given by

$$a_{m_1 \cdots m_n} = \frac{1}{m_1! \cdots m_n!} f^{(m_1 \cdots m_n)}(c_1, \ldots, c_n). \tag{1.16}$$

Proof. By replacing z_k by $z_k - c_k$, $k = 1, \ldots, n$, we may assume that $c_1 = c_2 = \cdots = c_n = 0$. For a point $z = (z_1, \ldots, z_n) \in U_\rho(0)$, $\rho = (\rho_1, \ldots, \rho_n)$, take $r = (r_1, \ldots, r_n)$ such that $|z_k| < r_k < \rho_k$ for $k = 1, \ldots, n$. Then $z \in U_r(0)$, and $[U_r(0)] \subset U_\rho(0) \subset D$. Hence by (1.10)

$$f(z) = \left(\frac{1}{2\pi i}\right)^n \int_{C^n} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \cdots d\zeta_n}{(\zeta_1 - z_1) \cdots (\zeta_n - z_n)}. \tag{1.17}$$

Since $|\zeta_k| = r_k > |z_k|$, $k = 1, \ldots, n$, we have

$$\frac{1}{\zeta_k - z_k} = \frac{1}{\zeta_k} \sum_{m_k=0}^{\infty} \left(\frac{z_k}{\zeta_k}\right)^{m_k}, \qquad \left|\frac{z_k}{\zeta_k}\right| = \frac{|z_k|}{r_k} < 1, \qquad k = 1, \ldots, n.$$

Substituting these into the right-hand side of (1.17), we obtain a power series expansion

$$f(z) = \sum_{m_1, \ldots, m_n = 0}^{\infty} a_{m_1 \cdots m_n} z_1^{m_1} \cdots z_n^{m_n},$$

$$a_{m_1 \cdots m_n} = \left(\frac{1}{2\pi i}\right)^n \int_{C^n} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \cdots d\zeta_n}{\zeta_1^{m_1+1} \cdots \zeta_n^{m_n+1}}.$$

Letting M be the maximum of $|f(\zeta_1, \ldots, \zeta_n)|$ on C^n, we have

$$|a_{m_1 \cdots m_n}| \leq \frac{M}{r_1^{m_1} \cdots r_n^{m_n}},$$

which proves that the above power series is absolutely convergent in $U_r(0)$. From (1.14), it is clear that $a_{m_1 \cdots m_n} = f^{(m_1 \cdots m_n)}(0)/m_1! \cdots m_n!$. ∎

(b) Power Series

In this section we consider a power series

$$P(z) = P(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n = 0}^{\infty} a_{m_1 \cdots m_n} z_1^{m_1} \cdots z_n^{m_n}$$

with centre 0. If $P(z)$ is a convergent at z, we denote its sum by the same notation $P(z)$.

Theorem 1.4. *Let $w = (w_1, \dots, w_n)$ be such that $w_1 \neq 0, \dots, w_n \neq 0$. If $P(z)$ is convergent at $z = w$, then $P(z)$ is absolutely convergent for $|z_1| < |w_1|, \dots, |z_n| < |w_n|$, and its sum $P(z)$ is a holomorphic function of n variables $z_1, \dots, z_n$ in $U_\rho(0)$ where $\rho = (|w_1|, \dots, |w_n|)$.*

Proof. For simplicity we consider the case $n = 2$. The general case is proved similarly. Since, by hypothesis, $P(z)$ is convergent, there exists a constant M such that $|a_{m_1 m_2} w_1^{m_1} w_2^{m_2}| \leqq M < +\infty$. Hence

$$|a_{m_1 m_2}| \leqq \frac{M}{\rho_1^{m_1} \rho_2^{m_2}},$$

where $\rho_1 = |w_1|$ and $\rho_2 = |w_2|$. Therefore if $|z_1| < \rho_1$ and $|z_2| < \rho_2$,

$$\sum_{m_1, m_2 = 0}^{\infty} |a_{m_1 m_2} z_1^{m_1} z_2^{m_2}| \leqq M \sum_{m_1 = 0}^{\infty} \left(\frac{|z_1|}{\rho_1}\right)^{m_1} \sum_{m_2 = 0}^{\infty} \left(\frac{|z_2|}{\rho_2}\right)^{m_2} < +\infty,$$

namely, $P(z)$ is absolutely convergent. Moreover taking arbitrary r_1 and r_2 with $0 < r_1 < \rho_1$, $0 < r_2 < \rho_2$, we have

$$|a_{m_1 m_2} z_1^{m_1} z_2^{m_2}| \leqq |a_{m_1 m_2}| r_1^{m_1} m_2^{m_2}, \qquad \sum_{m_1, m_2 = 0}^{\infty} |a_{m_1 m_2}| r_1^{m_1} r_2^{m_2} < +\infty$$

for $z = (z_1, z_2)$ with $|z_1| < r_1$ and $|z_2| < r_2$. Therefore $P(z)$ is uniformly and absolutely convergent in $[U_r(0)]$ with $r = (r_1, r_2)$, hence continuous in $[U_r(0)]$. Since r_1 and r_2 are arbitrary real numbers with $0 < r_1 < \rho_1$, $0 < r_2 < \rho_2$, and $P(z)$ is clearly a holomorphic function in z_1 and z_2 separately, $P(z_1, z_2)$ is holomorphic in $U_\rho(0)$. ∎

Replacing the variables z_k by $z_k - c_k$, $k = 1, \ldots, n$, we obtain a power series

$$P(z-c) = P(z_1 - c_1, \ldots, z_n - c_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1\cdots m_n}(z_1 - c_1)^{m_1}\cdots(z_n - c_n)^{m_n}$$

with centre $c = (c_1, \ldots, c_n)$.

Corollary. *If a power series $P(z-c)$ is convergent at $w = (w_1, \ldots, w_n)$ with $w_1 \neq c_1, \ldots, w_n \neq c_n$, $P(z-c)$ is absolutely convergent if $|z_k - c_k| < |w_k - c_k|$, $k = 1, \ldots, n$, and its sum $P(z-c)$ is a holomorphic function in $U_\rho(c)$, where $\rho = (|w_1 - c_1|, \ldots, |w_n - c_n|)$.* ∎

The region of convergence of a power series $P(z-c)$ is the union $D = \bigcup U_\rho(c)$ of all polydisks $U_\rho(c)$ where $P(z-c)$ is absolutely convergent. A region of convergence D is a domain if it is not empty. In case $n = 1$, the region of convergence of a power series is an empty set, an open disk, or the whole $\mathbb{C}$ itself, but in case $n \geqq 2$, the region of convergence of a power series may take various forms.

The next theorem follows immediately from this Corollary and Theorem 1.3.

Theorem 1.5. *A function $f(z) = f(z_1, \ldots, z_n)$ of n complex variables is holomorphic in a domain $D \subset \mathbb{C}^n$ if and only if for every point $c \in D$, $f(z)$ has a power series expansion $P(z-c)$ which is convergent in some neighbourhood of c.* ∎

(c) Cauchy-Riemann Equation

First consider a continuously differentiable function $f(z)$ of one complex variable z in a domain $D \subset \mathbb{C}$. Decompose z and $f(z)$ into their real and imaginary parts by writing $z = x + iy$ and $f(z) = u + iv$. Then u and v are continuously differentiable functions of the real coordinates x, y in D. Using z and $\bar{z}$, we have

$$x = \tfrac{1}{2}(z + \bar{z}), \qquad y = \frac{1}{2i}(z - \bar{z}).$$

Here z and $\bar{z}$ are *not* independent variables, but considering them as if they are independent, we define the partial derivatives of $f(z)$ with respect to z and $\bar{z}$ by

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right). \tag{1.18}$$

In terms of u and v, we have

$$\frac{\partial f}{\partial z}=\tfrac{1}{2}(u_x+v_y)+\frac{i}{2}(-u_y+v_x),$$
$$\frac{\partial f}{\partial \bar z}=\tfrac{1}{2}(u_x-v_y)+\frac{i}{2}(u_y+v_x). \tag{1.19}$$

Therefore by use of (1.18), the Cauchy-Riemann equation: $u_x=v_y$, $u_y=-v_x$, is written as

$$\frac{\partial f}{\partial \bar z}=0. \tag{1.20}$$

Thus a continuously differentiable function $f(z)$ is a holomorphic function of z in a domain D if and only if $\partial f/\partial \bar z=0$ identically in D. If $f(z)$ is holomorphic, $\partial f/\partial z=u_x+iv_x=f'(z)$ by (1.19), namely, for a holomorphic function $f(z)$, the partial derivative $(\partial f/\partial z)(z)$ is identical to the complex derivative $df(z)/dz$.

Next consider a function $f(z)=f(z_1,\dots,z_n)$ of n complex variables. Put $f(z)=u+iv$ as above. $f(z)$ is said to be continuously differentiable, C^r, C^∞, etc. if u and v are continuously differentiable, C^r, C^∞, etc. in the real coordinates $x_1,\dots,x_{2n}$.

Let $f(z)$ be a continuously differentiable function in a domain $W\subset\mathbb{C}^n$. Since

$$x_{2k-1}=\tfrac{1}{2}(z_k+\bar z_k),\qquad x_{2k}=\frac{1}{2i}(z_k-\bar z_k),\qquad k=1,\dots,n,$$

we have

$$\frac{\partial f}{\partial z_k}=\frac{1}{2}\left(\frac{\partial f}{\partial x_{2k-1}}-i\frac{\partial f}{\partial x_{2k}}\right),\qquad \frac{\partial f}{\partial \bar z_k}=\frac{1}{2}\left(\frac{\partial f}{\partial x_{2k-1}}+i\frac{\partial f}{\partial x_{2k}}\right). \tag{1.21}$$

Since $f(z)=f(z_1,\dots,z_n)$ is continuously differentiable, hence *a fortiori* continuous, $f(z)$ is a holomorphic function of n variables $z_1,\dots,z_n$ if and only if $f(z)$ is holomorphic in each z_k separately. Therefore from the above results, we obtain the following theorem.

Theorem 1.6. *Let $f(z)=f(z_1,\dots,z_n)$ be a continuously differentiable function of n complex variables $z_1,\dots,z_n$ in a domain $D\subset\mathbb{C}^n$. Then $f(z)$ is holomorphic in D if and only if*

$$\frac{\partial f}{\partial \bar z_k}=0,\qquad k=1,\dots,n. \quad \blacksquare \tag{1.22}$$

As is clear from Theorem 1.2, a holomorphic function $f(z)=f(z_1, \ldots, z_n)$ in $D \subset \mathbb{C}^n$ is a C^∞ function in the real coordinates $x_1, \ldots, x_{2n}$.

A differential operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{2n}^2}$$

is called a Laplacian, and a C^2-real function $u=u(x_1, \ldots, x_{2n})$ defined on a domain in $\mathbb{R}^{2n}$ is called a harmonic function if it satisfies the Laplace equation

$$\Delta u = 0.$$

Let $f(z)=u+iv$ be a holomorphic function of n complex variables $z_1, \ldots, z_n$. Then u and v are obviously harmonic functions since

$$\frac{\partial^2 u}{\partial x_{2k-1}^2} + \frac{\partial^2 u}{\partial x_{2k}^2} = 0, \qquad \frac{\partial^2 v}{\partial x_{2k-1}^2} + \frac{\partial^2 v}{\partial x_{2k}^2} = 0, \qquad k=1, \ldots, n.$$

Conversely in case $n=1$, let $u=u(x_1, x_2)$ be a harmonic function in a domain $D \subset \mathbb{C}$. Then for any $z \in D$, there is a holomorphic function $f(z)$ defined in some neighbourhood of z such that $u = \operatorname{Re} f(z)$ there. This does not hold in case $n \geqq 2$. For example, in case $n=2$, put $r^2 = x_1^2 + \cdots + x_4^2$. Then the function $u = 1/r^2$ is harmonic, but $\partial^2 u/\partial x_1^2 + \partial^2 u/\partial x_2^2 \neq 0$, hence, u cannot be the real part of any holomorphic function. This already reveals the essential difference between holomorphic functions of one variable, and those of n variables with $n \geqq 2$.

(d) Analytic Continuation

Theorem 1.7. *Let $f(x)=f(z_1, \ldots, z_n)$ be a holomorphic function in $D \subset \mathbb{C}^n$. Unless $f(z)$ is identically zero in D, at each point z of D, among $f(z)$ and all its partial derivatives $f^{(m_1 \cdots m_n)}(z)$ at least one does not vanish.*

Proof. Let D_0 be the set of points $z \in D$ such that $f(z)$ and all its partial derivatives vanish at z, and put $D_1 = D - D_0$. Then, $D = D_0 \cup D_1$, $D_1 \cap D_0 = \varnothing$ and D_0 is open by Theorem 1.3. Since D_1 is clearly open, and D is connected, either $D = D_0$ or $D = D_1$ holds. ∎

Corollary. *If two holomorphic functions $f(z)$ and $g(z)$ in a domain D coincide in some neighbourhood of a point $c \in D$, then $f(z)$ and $g(z)$ coincide in all of D.* ∎

As in the case of holomorphic functions of one complex variable, this Corollary implies the uniqueness of the analytic continuation.

First we must define the analytic continuation of a holomorphic function of n complex variables.

Let D_0 be a domain in $\mathbb{C}^n$, and $f_0(z)$ a holomorphic function in D_0.

Let D_1 be another domain in $\mathbb{C}^n$ with $D_0 \cap D_1 \neq \emptyset$. If there is a holomorphic function $f_1(z)$ defined on D_1 such that $f_1(z) = f_0(z)$ on $D_0 \cap D_1$, then $f_1(z)$ is said to be an *analytic continuation* of $f_0(z)$ *to* D_1. By the above Corollary, such $f_1(z)$ is unique if it exists.

Let $D_1, D_2, \ldots, D_n, \ldots$ be finitely or infinitely many domains in $\mathbb{C}^n$, and $f_k(z)$ a holomorphic function defined on each D_k. If for each $k \geqq 1$, $f_k(z)$ is an analytic continuation of $f_{k-1}(z)$, then every $f_k(z)$ is called an *analytic continuation* of $f_0(z)$. In this case putting $f(z) = f_k(z)$ for $z \in D_k$, we obtain a holomorphic function in $D = D_0 \cup D_1 \cup \cdots$. If this $f(z)$ is one-valued in D, then $f(z)$ is an analytic continuation of $f_0(z)$ to D in the above sense. In general, however, $f(z)$ is not necessarily one-valued, but in these cases too, we call $f(z)$ an *analytic continuation* of $f_0(z)$.

As in the case of functions of one variable, by saying simply a holomorphic function, we mean a one-valued holomorphic function, whereas a holomorphic function which may not be one-valued is called an analytic function.

Unlike in the case $n=1$, in case $n \geqq 2$, there is a domain $D_0 \subset \mathbb{C}^n$ for which there exists a domain $D \supsetneqq D_0$ such that every holomorphic function defined on D_0 can be continued analytically to D. To see this, first consider the case $n=2$. Let $\rho_1, \rho_2, \sigma_1, \sigma_2$ be real numbers with $0 < \sigma_1 < \rho_1$, $0 < \sigma_2 < \rho_2$, and let D be a polydisk with centre 0:

$$D = U_\rho(0) = \{(z_1, z_2) \,|\, |z_1| < \rho_1, |z_2| < \rho_2\}, \qquad \rho = (\rho_1, \rho_2).$$

Put

$$T(0) = \{(z_1, z_2) \,|\, \mathrm{Re}\, z_1 \geqq -\sigma_1, |z_2| \leqq \sigma_2\},$$

and $D_0 = D - T(0)$. If we write $z_1 = x_1 + ix_2$, and $z_2 = x_3 + ix_4$, the section of D_0 by the hyperplane $x_2 = 0$ is illustrated in Fig. 1.

Every holomorphic function defined on $D_0 = D - T(0)$ has an analytic continuation to D.

Proof. Let γ_r: $\theta \to \gamma_r(\theta) = (z_1, re^{i\theta})$, $0 \leqq \theta \leqq 2\pi$, be the circle of radius r with centre $(z_1, 0) \in D$, where $\sigma_2 < r < \rho_2$. Then $\gamma_r \subset D_0$. Consider the integral along γ_r

$$f_1(z_1, z_2) = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f_0(z_1, \zeta)}{\zeta - z_2} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} \frac{f_0(z_1, re^{i\theta})}{1 - r^{-1} e^{-i\theta} z_2} d\theta \qquad (1.23)$$