

Sau Fa Kwok

Langevin and Fokker-Planck Equations and their Generalizations

Descriptions and Solutions



 World Scientific

Langevin and Fokker-Planck Equations and their Generalizations

Descriptions and Solutions

This invaluable book provides a broad introduction to a rapidly growing area of nonequilibrium statistical physics. The first part of the book complements the classical book on the Langevin and Fokker-Planck equations (H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer, 1996)). Some topics and methods of solutions are presented and discussed in details which are not described in Risken's book, such as the method of similarity solution, the method of characteristics, transformation of diffusion processes into the Wiener process in different prescriptions, harmonic noise and relativistic Brownian motion. Connection between the Langevin equation and Tsallis distribution is also discussed.

Due to the growing interest in the research on the generalized Langevin equations, several of them are presented. They are described with some details.

Recent research on the integro-differential Fokker-Planck equation derived from the continuous time random walk model shows that the topic has several aspects to be explored. This equation is worked analytically for the linear force and the generic waiting time probability distribution function. Moreover, generalized Klein-Kramers equations are also presented and discussed. They have the potential to be applied to natural systems, such as biological systems.

World Scientific

www.worldscientific.com

9745 hc

ISBN 978-981-3228-40-5



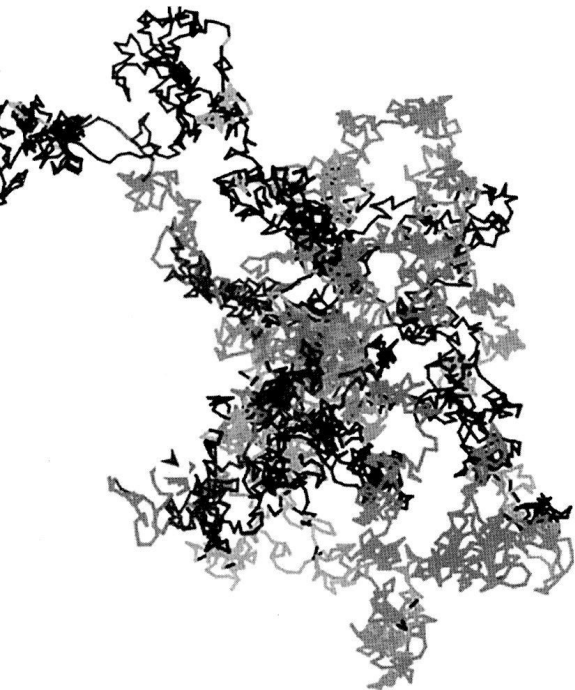
9 789813 228405

Kwok



Langvin and Fokker-Planck Equations and their Generalizations





Langevin and Fokker-Planck Equations and their Generalizations

Descriptions and Solutions

Sau Fa Kwok

State University of Maringá, Brazil

 **World Scientific**

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI • TOKYO

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Names: Kwok, Sau Fa, author.

Title: Langevin and Fokker-Planck equations and their generalizations :

descriptions and solutions / by Sau Fa Kwok (State University of Maringá, Brazil).

Description: New Jersey : World Scientific, 2017. | Includes bibliographical references and index.

Identifiers: LCCN 2017042757 | ISBN 9789813228405 (hardcover : alk. paper)

Subjects: LCSH: Statistical mechanics. | Differential equations, Partial. | Statistical physics.

Classification: LCC QC174.8 .K96 2017 | DDC 530.13--dc23

LC record available at <https://lcn.loc.gov/2017042757>

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Copyright © 2018 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

For any available supplementary material, please visit

<http://www.worldscientific.com/worldscibooks/10.1142/9745#t=suppl>

Printed in Singapore

**Langevin and
Fokker-Planck Equations
and their Generalizations**

Descriptions and Solutions

This book is dedicated to the memory of my grandfather who illuminated
my way, bringing inspiration for my life journey.

Preface

This book deals with some mathematical formulations of the nonequilibrium statistical physics: The Langevin equation, the Fokker-Planck equation, the continuous time random walk model and their generalizations. Despite the origin, they have been employed to describe nonequilibrium systems in different areas, such as physics, biology, chemistry, hydrology and economics.

The first part of the book complements the classical book on the Langevin and Fokker-Planck equations (H. Risken, *The Fokker-Planck equation*, 1996). Due to the growing interest in the researches on the generalized Langevin equations, several of them are presented; they are described with some details. The last part is devoted to the continuous time random walk model (CTRW). The book is sketched as follows. Chapter 2 is dedicated to the derivation of the Fokker-Planck equation (for one dimension) from the Langevin equation for different orders of prescription in discretization rules for the stochastic integrals due to the multiplicative white noise. Chapter 3 deals with various methods for solving the Fokker-Planck equation which are not described in the Risken's book, such as the method of similarity solution, the method of characteristics, transformations of diffusion processes into the Wiener process for different orders of prescription in discretization rules for the stochastic integrals, and colored noise. Connection between the Langevin equation and Tsallis distribution is also discussed. Chapter 4 describes the Fokker-Planck equation for several variables which includes the relativistic Brownian motion. Chapter 5 introduces and discusses some generalized Langevin equations which contain non-local operators in time. Chapters 6 and 7 deal with the continuous time random walk model and the derivation of integro-differential Fokker-Planck equation; derivation of an integro-differential Klein-Kramers

equation from the generalized Chapman-Kolmogorov equation; methods of solutions; generalization of the integro-differential Klein-Kramers equation for the description of superdiffusive regime.

Finally, I would like to thank Dr. Swee Cheng Lim and Dr. Ke Gang Wang for their suggestions and co-operation.

For comments and suggestions, please, send to the email: kwok@dfi.uem.br.

Maringá, May, 2017

Kwok Sau Fa

Contents

<i>Preface</i>	vii
1. Introduction	1
2. Langevin and Fokker-Planck equations	5
2.1 Introduction	5
2.2 General Langevin equation for one variable	5
2.3 Kramers-Moyal expansion coefficients	7
2.4 Ito, Stratonovich and other prescriptions	9
2.5 Kramers-Moyal expansion and Fokker-Planck equation . .	12
2.6 General Langevin equation for several variables	14
2.7 Appendices	16
2.7.1 Colored noise	16
3. Fokker-Planck equation for one variable and its solution	21
3.1 Introduction	21
3.2 Time-independent drift and diffusion coefficients	22
3.2.1 Stationary solution	24
3.2.2 Method of separation of variables and eigenfunction expansion	25
3.2.3 Solution for the harmonic potential	26
3.3 Solution by the method of transformation of variables . .	28
3.3.1 Solution for constant diffusion coefficient and the general linear force	29
3.3.2 Solution for time dependent drift coefficient and variable diffusion coefficient	33

3.4	Langevin equation with multiplicative white noise: Transformation of diffusion processes into the Wiener process in different prescriptions	36
3.4.1	Deterministic drift $h_1(x, t)$ and multiplicative noise term $h_2(x, t)$ are separable in time and space . . .	36
3.4.2	General drift $h_1(x, t)$ and multiplicative noise term $h_2(x, t)$	39
3.4.3	Applications	41
3.5	Similarity solution	49
3.5.1	Similarity solution	49
3.6	Solution in a finite interval and first passage time	53
3.6.1	First passage time distribution and mean first passage time	53
3.6.2	Solution for vanishing drift coefficient and constant diffusion coefficient	54
3.6.3	Solution for power-law diffusion coefficient	56
3.7	Langevin equation with multiplicative noise in different orders of prescription and its connection with the Tsallis distribution	59
3.7.1	Tsallis distribution and the atom-laser interaction in the optical lattice	61
3.7.2	Tsallis distribution and a class of population growth models with linearly coupled noise	62
4.	Fokker-Planck equation for several variables	67
4.1	Introduction	67
4.2	Ornstein-Uhlenbeck process	68
4.3	The Klein-Kramers equation for a linear force in a one-dimensional space	73
4.4	Harmonic oscillator driven by colored noise in a one-dimensional space	77
4.4.1	Stationary solution	78
4.4.2	Time-dependent solution	81
4.5	Fokker-Planck equations of the relativistic Brownian motion	88
4.6	Appendices	92
4.6.1	Numerical inversion of Laplace transforms	92
5.	Generalized Langevin equations	95

5.1	Introduction	95
5.2	Derivation of the generalized Langevin equation from the Hamiltonian formalism	96
5.2.1	General analysis: First two moments, variances, covariance, PDF and Fokker-Planck equation . . .	98
5.3	Fractional Langevin equation	106
5.4	Appendices	112
5.4.1	Generalized Mittag-Leffler function	112
6.	Continuous Time Random Walk model	117
6.1	Introduction	117
6.2	Uncoupled Continuous Time Random Walk model	120
6.3	Integro-differential equations	121
6.3.1	Integro-differential diffusion equation and Integro-differential Fokker-Planck equation	122
6.3.2	Generalized Chapman-Kolmogorov equation and integro-differential Klein-Kramers equation	125
6.4	Appendices	130
6.4.1	Integro-differential diffusion equation with external force	130
6.4.2	Riesz space fractional derivative	131
7.	Uncoupled Continuous Time Random Walk model and its solution	137
7.1	Introduction	137
7.2	Integro-differential diffusion equation for force-free	137
7.2.1	The first two moments and PDF	137
7.2.2	First passage time density and mean first passage time	147
7.2.3	Full decoupled case	152
7.3	Integro-differential Fokker-Planck equation or integro-differential diffusion equation with external force	162
7.3.1	Correlation function, mean square displacement, intermediate scattering function and dynamic structure factor	166
7.4	Generalized Rayleigh equation	176
7.5	Integro-differential Klein-Kramers equation	178
7.6	Generalized Klein-Kramers equation	181

7.6.1	First two moments for velocity and displacement in the force-free case	184
7.6.2	Solution of the generalized Klein-Kramers equation	186
7.7	Applications	188
7.8	Appendices	188
7.8.1	Wright function	188
<i>Index</i>		193

Chapter 1

Introduction

Diffusion is a very common process in nature and it can be observed in many different systems. In particular, anomalous diffusion processes have been observed in most systems investigated such as bacterial cytoplasm motion [1], conformational fluctuations within a single protein molecule [2, 3], fluorescence intermittency in single enzymes [4], the cell migration of two migrating transformed renal epithelial MadinDarby canine kidney (MDCK-F) cell strains [5], the internal protein dynamics for the backbone atoms of hydrated elastin [6] and white and gray matters of a fixed rat brain [7]. Their properties have also been extensively investigated by various approaches in order to model different kinds of probability distributions such as long-range spatial or temporal correlations [8–14]. For instance, we may cite the Langevin equation [15–18], the Fokker-Planck equation [15–17], the continuous-time random walk model (CTRW) [8, 19], the generalized Langevin equation [20–22], the fractional Fokker-Planck equation [8] and the nonlinear Fokker-Planck equation [23]. These approaches have also been used to describe numerous systems in various contexts, such as economics, physics, hydrology, chemistry and biology. The diffusion process is classified according to the mean square displacement (MSD)

$$\langle x^2(t) \rangle \sim t^\alpha. \quad (1.1)$$

In the case of normal diffusion, the MSD grows linearly with time ($\alpha = 1$). For $0 < \alpha < 1$ the process is called subdiffusive, and for $\alpha > 1$ the process is called superdiffusive. For $\alpha = 2$ it is referred to as ballistic motion. The well-established property of the normal diffusion described by the Gaussian distribution (the probability distribution is denoted by $\rho(x, t)$, and $\rho(x, t)dx$ is the probability for finding a particle in a position between x and $x + dx$

at time t) can be obtained from the ordinary Langevin equation

$$\frac{d\xi}{dt} = \sqrt{D}L(t), \quad (1.2)$$

where ξ is a stochastic variable, D is the diffusion coefficient and $L(t)$ is the Langevin force which is assumed to be a Gaussian random variable, or from the Fokker-Planck equation

$$\frac{\partial \rho(x, t)}{\partial t} = -\frac{\partial}{\partial x} [D_1(x, t)\rho(x, t)] + \frac{\partial^2}{\partial x^2} [D_2(x, t)\rho(x, t)] \quad (1.3)$$

with vanishing drift coefficient $D_1(x, t) = 0$ and constant diffusion coefficient $D_2(x, t) = D$ [15, 16]; it can also be obtained from an integro-differential diffusion equation

$$\frac{\partial \rho(x, t)}{\partial t} - \int_0^t dt_1 g(t - t_1) \frac{\partial \rho(x, t_1)}{\partial t_1} = C \frac{\partial}{\partial t} \int_0^t dt_1 g(t - t_1) \frac{\partial^2 \rho(x, t_1)}{\partial x^2} \quad (1.4)$$

with the exponential function for the waiting time probability distribution, $g(t) = be^{-bt}$ [24]. Anomalous diffusion regimes can also be obtained from the ordinary Fokker-Planck equation, however, they arise from a variable diffusion coefficient which may depend on time and/or space. Besides, in the view of the Langevin approach it may be associated with a multiplicative noise term. In the case of generalized Langevin equation [20, 21], it is described by

$$\frac{dv}{dt} + \int_0^t dt_1 \gamma(t - t_1) v(t_1) = F(x) + Q(t) \quad (1.5)$$

with a unitary mass (where $F(x)$ and $Q(t)$ are deterministic and stochastic forces, respectively).

In other approaches, such as the fractional Langevin equations and generalized Fokker-Planck equations (fractional and nonlinear) [23, 25–28], they can also describe anomalous diffusion processes.

This book provides a broad introduction to a rapidly growing area of nonequilibrium statistical physics. The first part of the book complements the classical book on the Langevin and Fokker-Planck equations [15]. Some topics and methods of solutions are presented and discussed in details which are not described in Ref. [15], such as the method of similarity solution, the method of characteristics, transformations of diffusion processes into the Wiener process for different orders of prescription in discretization rules for the stochastic integrals, and harmonic noise.

Due to the growing interest in the research on the generalized Langevin equations, several of them are presented. They are described with some details.