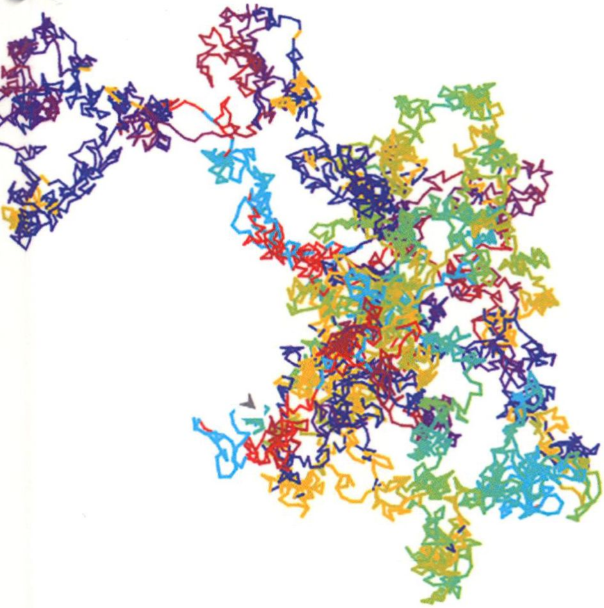

Sau Fa Kwok

Langevin and Fokker-Planck Equations and their Generalizations

Descriptions and Solutions

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Langevin and Fokker-Planck Equations and their Generalizations

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This invaluable book provides a broad introduction to a rapidly growing area of nonequilibrium statistical physics. The first part of the book complements the classical book on the Langevin and Fokker–Planck equations (H. Risken, The Fokker–Planck Equation: Methods of Solution and Applications (Springer, 1996)). Some topics and methods of solutions are presented and discussed in details which are not described in Risken's book, such as the method of similarity solution, the method of characteristics, transformation of diffusion processes into the Wiener process in different prescriptions, harmonic noise and relativistic Brownian motion. Connection between the Langevin equation and Tsallis distribution is also discussed.

Due to the growing interest in the research on the generalized Langevin equations, several of them are presented. They are described with some details.

Recent research on the integro-differential Fokker–Planck equation derived from the continuous time random walk model shows that the topic has several aspects to be explored. This equation is worked analytically for the linear force and the generic waiting time probability distribution function. Moreover, generalized Klein-Kramers equations are also presented and discussed. They have the potential to be applied to natural systems, such as biological systems.

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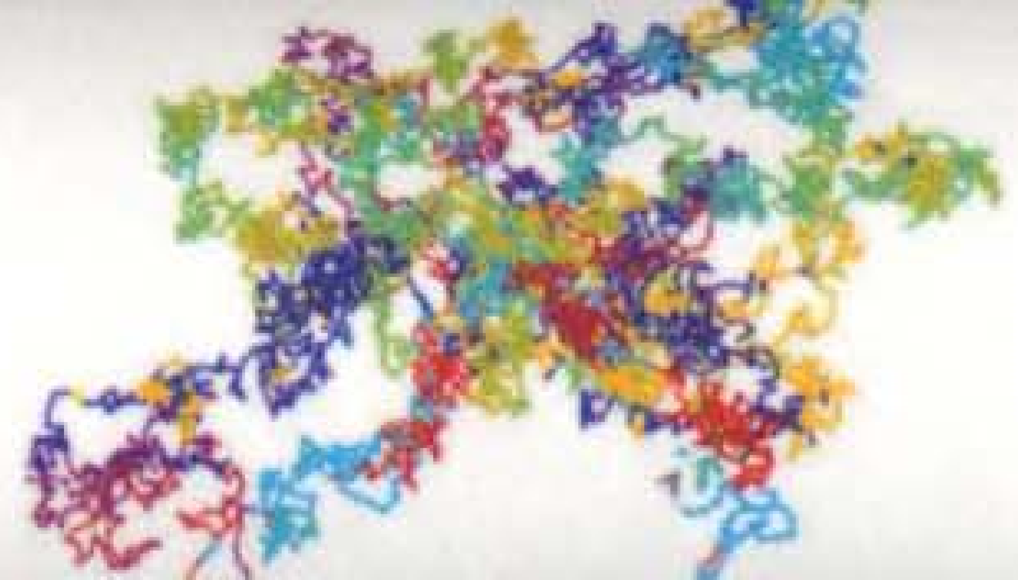

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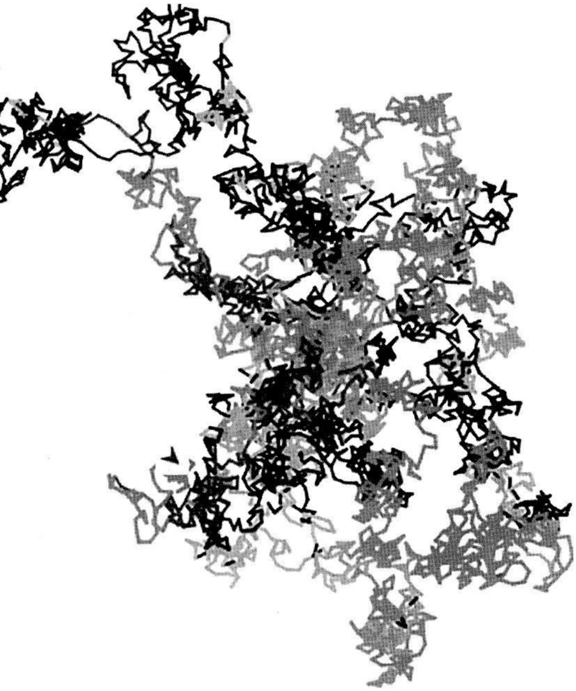

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Langevin and Fokker-Planck Equations and their Generalizations

Descriptions and Solutions

This book is dedicated to the memory of my grandfather who illuminated my way, bringing inspiration for my life journey.

Preface

This book deals with some mathematical formulations of the nonequilibrium statistical physics: The Langevin equation, the Fokker-Planck equation, the continuous time random walk model and their generalizations. Despite the origin, they have been employed to describe nonequilibrium systems in different areas, such as physics, biology, chemistry, hydrology and economics.

The first part of the book complements the classical book on the Langevin and Fokker-Planck equations (H. Risken, The Fokker-Planck equation, 1996). Due to the growing interest in the researches on the generalized Langevin equations, several of them are presented; they are described with some details. The last part is devoted to the continuous time random walk model (CTRW). The book is sketched as follows. Chapter 2 is dedicated to the derivation of the Fokker-Planck equation (for one dimension) from the Langevin equation for different orders of prescription in discretization rules for the stochastic integrals due to the multiplicative white noise. Chapter 3 deals with various methods for solving the Fokker-Planck equation which are not described in the Risken's book, such as the method of similarity solution, the method of characteristics, transformations of diffusion processes into the Wiener process for different orders of prescription in discretization rules for the stochastic integrals, and colored noise. Connection between the Langevin equation and Tsallis distribution is also discussed. Chapter 4 describes the Fokker-Planck equation for several variables which includes the relativistic Brownian motion. Chapter 5 introduces and discusses some generalized Langevin equations which contain non-local operators in time. Chapters 6 and 7 deal with the continuous time random walk model and the derivation of integro-differential Fokker-Planck equation; derivation of an integro-differential Klein-Kramers equation from the generalized Chapman-Kolmogorov equation; methods of solutions; generalization of the integro-differential Klein-Kramers equation for the description of superdiffusive regime.

Finally, I would like to thank Dr. Swee Cheng Lim and Dr. Ke Gang Wang for their suggestions and co-operation.

For comments and suggestions, please, send to the email: kwok@dfi.uem.br.

Maringá, May, 2017 *Kwok Sau Fa*

Contents

Chapter 1

Introduction

Diffusion is a very common process in nature and it can be observed in many different systems. In particular, anomalous diffusion processes have been observed in most systems investigated such as bacterial cytoplasm motion [1], conformational fluctuations within a single protein molecule [2, 3], fluorescence intermittency in single enzymes [4], the cell migration of two migrating transformed renal epithelial MadinDarby canine kidney (MDCK-F) cell strains [5], the internal protein dynamics for the backbone atoms of hydrated elastin [6] and white and gray matters of a fixed rat brain [7]. Their properties have also been extensively investigated by various approaches in order to model different kinds of probability distributions such as long-range spatial or temporal correlations [8–14]. For instance, we may cite the Langevin equation [15–18], the Fokker-Planck equation [15–17], the continuous-time random walk model (CTRW) [8, 19], the generalized Langevin equation [20–22], the fractional Fokker-Planck equation [8] and the nonlinear Fokker-Planck equation [23]. These approaches have also been used to describe numerous systems in various contexts, such as economics, physics, hydrology, chemistry and biology. The diffusion process is classified according to the mean square displacement (MSD)

$$\langle x^2(t)\rangle \sim t^{\alpha}. \tag{1.1}$$

In the case of normal diffusion, the MSD grows linearly with time ($\alpha = 1$). For $0 < \alpha < 1$ the process is called subdiffusive, and for $\alpha > 1$ the process is called superdiffusive. For $\alpha = 2$ it is referred to as ballistic motion. The well-established property of the normal diffusion described by the Gaussian distribution (the probability distribution is denoted by $\rho(x,t)$, and $\rho(x,t)dx$ is the probability for finding a particle in a position between x and $x + dx$

at time t) can be obtained from the ordinary Langevin equation

$$\frac{d\xi}{dt} = \sqrt{D}L(t), \tag{1.2}$$

where ξ is a stochastic variable, D is the diffusion coefficient and $L(t)$ is the Langevin force which is assumed to be a Gaussian random variable, or from the Fokker-Planck equation

$$\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[D_1(x,t)\rho(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D_2(x,t)\rho(x,t)\right] \tag{1.3}$$

with vanishing drift coefficient $D_1(x,t) = 0$ and constant diffusion coefficient $D_2(x,t) = D$ [15, 16]; it can also be obtained from an integro-differential diffusion equation

$$\frac{\partial\rho(x,t)}{\partial t} - \int_0^t \mathrm{d}t_1 g\left(t-t_1\right)\frac{\partial\rho(x,t_1)}{\partial t_1} = C\frac{\partial}{\partial t}\int_0^t \mathrm{d}t_1 g\left(t-t_1\right)\frac{\partial^2\rho(x,t_1)}{\partial x^2} \tag{1.4}$$

with the exponential function for the waiting time probability distribution, $g(t) = be^{-bt}$ [24]. Anomalous diffusion regimes can also be obtained from the ordinary Fokker-Planck equation, however, they arise from a variable diffusion coefficient which may depend on time and/or space. Besides, in the view of the Langevin approach it may be associated with a multiplicative noise term. In the case of generalized Langevin equation [20, 21], it is described by

$$\frac{dv}{dt} + \int_0^t \mathrm{d}t_1\gamma\left(t-t_1\right)v\left(t_1\right) = F(x) + Q(t) \tag{1.5}$$

with a unitary mass (where $F(x)$ and $Q(t)$ are deterministic and stochastic forces, respectively).

In other approaches, such as the fractional Langevin equations and generalized Fokker-Planck equations (fractional and nonlinear) [23,25–28], they can also describe anomalous diffusion processes.

This book provides a broad introduction to a rapidly growing area of nonequilibrium statistical physics. The first part of the book complements the classical book on the Langevin and Fokker-Planck equations [15]. Some topics and methods of solutions are presented and discussed in details which are not described in Ref. [15], such as the method of similarity solution, the method of characteristics, transformations of diffusion processes into the Wiener process for different orders of prescription in discretization rules for the stochastic integrals, and harmonic noise.

Due to the growing interest in the research on the generalized Langevin equations, several of them are presented. They are described with some details.