Metasolutions of Parabolic Equations in Population Dynamics

Julián López-Gómez

$$\frac{\partial u}{\partial t} - d\,\Delta u = \lambda u + a(x)\,u^2$$

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Metasolutions of Parabolic Equations in Population Dynamics

To Rosa Gómez-González
my mother,
with love and gratitude

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Preface

This book studies the dynamics of a generalized prototype of the semilinear parabolic logistic problem

$$\begin{cases} \frac{\partial u}{\partial t} - d\Delta u = \lambda u + a(x)u^2 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0 > 0 & \text{in } \Omega, \end{cases} \tag{P.1}$$

where $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, is a smooth bounded domain, $t > 0$ stands for the time, and $a(x)$ is an arbitrary continuous function such that $a(x) \leq 0$ for all $x \in \Omega$ but $a \neq 0$. So, (P.1) is a parabolic boundary value problem for the *degenerate diffusive logistic equation*

$$\frac{\partial u}{\partial t} - d\Delta u = \lambda u + a(x)u^2 \tag{P.2}$$

in Ω. It is said to be *degenerate* because $a(x)$ can vanish on some patches of Ω, in contrast to the classical case when $a(x) < 0$ for all $x \in \bar{\Omega}$.

In the context of population dynamics, $N \leq 3$, Ω is the inhabiting area where the individuals of a species, u, disperse randomly at a constant rate measured by $d > 0$; $u(x,t)$ is the density of the individuals of the species at the location $x \in \Omega$ after time $t > 0$; λ is the *intrinsic rate of natural increase* of the species; u_0 is the initial distribution of the species in Ω; and

$$K(x) \equiv -\frac{\lambda}{a(x)}, \qquad x \in \Omega, \tag{P.3}$$

is the *carrying capacity* of Ω at each location $x \in \Omega$. As we are imposing homogeneous Dirichlet boundary conditions on $\partial\Omega$, the surroundings of Ω are assumed to be hostile for the species u. So, no individual of the species can survive on the habitat edges. However, this assumption is far from necessary for the validity of most of the results discussed in this book. Two classic books on population dynamics from the perspective of reaction diffusion equations are by J. D. Murray [197] and A. Okubo and S. A. Levin [200].

Although it is folklore that the classical non-spatial logistic equation

$$u'(t) = \lambda u(t) + au^2(t)$$

where a is a negative constant goes back to P. F. Verhulst [230] (1838), it is less known that the diffusive logistic equation (P.2) was introduced by A. N.

Kolmogorov I. G. Petrovsky and N. S. Piskunov [124], and independently by R. A. Fisher [88], in 1937, to study some problems of a biological nature. In the classical context, $a(x)$ is a continuous function such that $a(x) < 0$ for all $x \in \bar{\Omega}$. The analysis of the degenerate parabolic problem when $a \leq 0$ in Ω but $a \equiv 0$ on some subset of Ω with non-empty interior goes back to J. M. Fraile et al. [90] (1996). An elliptic counterpart of these degenerate models had been previously analyzed by H. Brézis and L. Oswald [31], and by T. Ouyang [203], [204], as part of his PhD thesis under the supervision of W. M. Ni.

Naturally, in spatially heterogeneous environments, the carrying capacity, $K(x)$, might suffer dramatic variations according to the location of the individuals of the species on the territory, $x \in \Omega$. Indeed, although $K(x)$ might be very small on some patches of the territory as an effect of harsh environmental conditions or abiotic stress, in benign areas, natural refuges or special protected zones, $K(x)$ might reach huge values.

From the mathematical point of view, a rather reasonable methodology to deal with huge variations of the carrying capacity $K(x)$ in the territory Ω is assuming that $K = \infty$, or equivalently $a \equiv 0$, in the 'protected areas,' while it is finite in less favorable zones. This strategy also makes sense from the biological point of view, as it is equivalent to combining, simultaneously, within the same territory, the Mathus and the Verhulst laws regulating the growth of the species. A further perturbation analysis should reveal the complete list of admissible limiting distribution patterns of the population as time passes in general diffusive spatially heterogeneous logistic problems.

In the region where $a(x) < 0$ the temporal evolution of species u is assumed to be governed by a logistic growth, while in the region where $a(x) \equiv 0$ the species u increases according to an exponential growth. The main goal of this book is predicting the time evolution of the species u in Ω under such circumstances. Should the species exhibit a genuine logistic behavior in Ω, or, on the contrary, should it exhibit an exponential growth? There is the possibility that u grows according to the Malthus law on some areas of Ω, while it simultaneously inherits a limited growth on others.

In an effort to summarize the contents of this book in this short general preliminary presentation, suppose $a(x)$ has a nodal behavior of the type described in Figure P.1, where the territory Ω contains ten *protected zones*, $\Omega^1_{0,1}$, $\Omega^2_{0,1}$, which are two balls, or discs if $N = 2$, with the same radius R_1, $\Omega^1_{0,2}$, $1 \leq i \leq 4$, which are four balls with radius $R_2 < R_1$, and $\Omega^i_{0,3}$, $1 \leq i \leq 4$, which are four balls with radius $R_3 < R_2$. The weight function $a(x)$ is assumed to vanish in all these *refuges*, or protected zones, while it is negative on their complement, the shadow region of Figure P.1, denoted by Ω_-.

To describe the main findings of this book for this special configuration of the territory we need to introduce some notation. Given any nice open connected subset, D, of Ω we will denote by $\lambda_1[-\Delta, D]$ the lowest eigenvalue of the linear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } D, \\ u = 0 & \text{on } \partial D. \end{cases} \tag{P.4}$$

As will be discussed in Chapter 1, from a biological perspective, $d\lambda_1[-\Delta, D]$ measures the critical size of the rate of natural increase, λ, so that the inhabited area D can maintain the species u dispersing at the rate d in the patch D, in the sense that u is driven to extinction if $\lambda < d\lambda_1[-\Delta, D]$, while it is permanent if $\lambda > d\lambda_1[-\Delta, D]$. So, the condition $d\lambda_1[-\Delta, D] < \lambda$ measures the necessary geometrical properties and size of the patch D to maintain the species dispersing at the rate d in D with an intrinsic rate of natural increase λ. When D is a ball of radius R, a simple change of scale reveals that

$$\lambda_1[-\Delta, D] = \frac{\lambda_1[-\Delta, B_1]}{R^2}$$

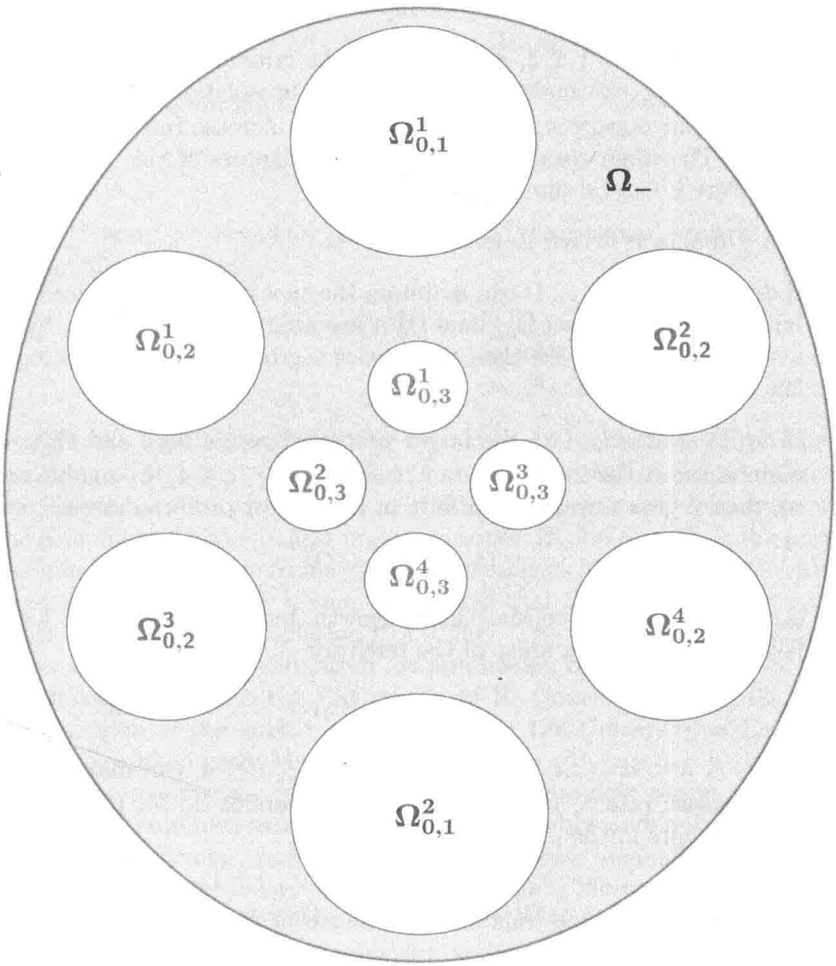

FIGURE P.1: An admissible nodal configuration for $a(x)$.

where B_1 is the unit ball of $\mathbb{R}^N$. In particular, the larger the protected zone the smaller the principal eigenvalue. Consequently, setting

$$\sigma_0 = \lambda_1[-\Delta, \Omega], \qquad \sigma_j := \lambda_1[-\Delta, \Omega^1_{0,j}], \qquad 1 \le j \le 3,$$

it becomes apparent that

$$\begin{aligned} d\sigma_0 = d\lambda_1[-\Delta, \Omega] < d\sigma_1 = d\lambda_1[-\Delta, \Omega^i_{0,1}] = \frac{d\lambda_1[-\Delta, B_1]}{R_1^2} \quad (1 \le i \le 2) \\ < d\sigma_2 = d\lambda_1[-\Delta, \Omega^i_{0,2}] = \frac{d\lambda_1[-\Delta, B_1]}{R_2^2} \quad (1 \le i \le 4) \\ < d\sigma_3 = d\lambda_1[-\Delta, \Omega^i_{0,3}] = \frac{d\lambda_1[-\Delta, B_1]}{R_3^2} \quad (1 \le i \le 4) \end{aligned}$$

Naturally, for each $j = 1, 2, 3$, $d\sigma_j$ measures the critical size of λ so that the protected zone $\Omega^i_{0,j}$ can maintain the species u in isolation. In other words, $\Omega^i_{0,j}$ has sufficient resources to maintain u at the increase rate λ if, and only if, $\lambda > d\sigma_j$. The main results of the first five chapters of this book, which constitute Part I, can be summarized as follows:

- If $\lambda \le d\sigma_0$, u is driven to extinction in Ω.

- If $d\sigma_0 < \lambda < d\sigma_1$, i.e., Ω can maintain the species at the increase rate λ, but the larger refuges, $\Omega^1_{0,1}$ and $\Omega^2_{0,1}$, are unable to maintain it, by e.g., a shortage of resources, then the species u grows according to a logistic law everywhere in Ω.

- If $d\sigma_1 \le \lambda < d\sigma_2$, i.e., the larger protected zones, $\Omega^1_{0,1}$ and $\Omega^2_{0,1}$, can maintain u at the increase rate λ, but $\Omega^i_{0,2}$, $1 \le i \le 4$, are unable to do so, then u grows up exponentially in the largest protected areas

  $$\Omega_{0,1} \equiv \Omega^1_{0,1} \cup \Omega^2_{0,1},$$

  according to a genuine Malthusian growth, but according to the logistic law in the remaining areas of the territory

  $$\Omega_1 \equiv \Omega \setminus \bar{\Omega}_{0,1}.$$

- If $d\sigma_2 \le \lambda < d\sigma_3$, i.e., the refuges $\Omega^i_{0,2}$, $1 \le i \le 4$, can maintain u at the increase rate λ, but $\Omega^i_{0,3}$, $1 \le i \le 4$, cannot do so, then u grows exponentially in the protected areas

  $$\Omega_{0,1} \equiv \Omega^1_{0,1} \cup \Omega^2_{0,1} \quad \text{and} \quad \Omega_{0,2} \equiv \bigcup_{i=1}^{4} \Omega^i_{0,1}$$

  whereas it has a limited logistic increase in

  $$\Omega_2 \equiv \Omega \setminus (\bar{\Omega}_{0,1} \cup \bar{\Omega}_{0,2}).$$

- If $\lambda \geq d\sigma_3$, i.e., any refuge is able to maintain u at the increase rate λ, then u grows exponentially in all the protected zones

$$\Omega_{0,1} \equiv \Omega_{0,1}^1 \cup \Omega_{0,1}^2, \quad \Omega_{0,2} \equiv \bigcup_{i=1}^{4} \Omega_{0,2}^i \quad \text{and} \quad \Omega_{0,3} \equiv \bigcup_{i=1}^{4} \Omega_{0,3}^i$$

but according to a logistic law in the region

$$\Omega_3 \equiv \Omega \setminus (\bar{\Omega}_{0,1} \cup \bar{\Omega}_{0,2} \cup \bar{\Omega}_{0,3}) = \Omega_-.$$

Consequently, if, for instance, $d\sigma_1 < \lambda < d\sigma_2$, and we denote by $u(x,t)$ the unique solution of (P.1), then, as a consequence of the analysis in Part I, we find that

$$\lim_{t\uparrow\infty} u(x,t) = \infty \quad \text{for all} \quad x \in \Omega_{0,1} = \Omega_{0,1}^1 \cup \Omega_{0,1}^2$$

whereas, in the region $\Omega \setminus \bar{\Omega}_{0,1}$,

$$L_\lambda^{\min} \leq \liminf_{t\to\infty} u(\cdot,t) \leq \limsup_{t\to\infty} u(\cdot,t) \leq L_\lambda^{\max} \tag{P.5}$$

where $L_\lambda^{\min}$ and $L_\lambda^{\max}$ stand for the minimal and the maximal positive solutions of the singular problem

$$\begin{cases} -d\Delta L = \lambda L + a(x)L^2 & \text{in } \Omega \setminus \bar{\Omega}_{0,1}, \\ L = \infty & \text{on } \partial\Omega_{0,1}^1 \cup \partial\Omega_{0,1}^2, \\ L = 0 & \text{on } \partial\Omega. \end{cases} \tag{P.6}$$

Therefore, the limiting profile of $u(x,t)$ as time $t \uparrow \infty$ becomes infinity in the larger refuges, $\bar{\Omega}_{0,1}^1$ and $\bar{\Omega}_{0,1}^2$, while it remains bounded in the complement. These limiting profiles are referred to in this book as *metasolutions* supported in the complement of the largest protected zones, Ω_1, because Ω_1 is the portion of the inhabiting area where the growth of u inherits a genuine logistic character and hence it is limited. It should be noted that the smaller refuges cannot support the species u in isolation if $\lambda < d\sigma_2$. The formal concept metasolution was coined in [109], submitted for publication in September 1998. Then, it was incorporated into the PhD thesis of R. Gómez-Reñasco [105], under the supervision of the author and defended at the University of La Laguna (Tenerife, Spain) in early May 1999.

For those readers not familiarized yet with the most recent advances in the theory of nonlinear parabolic problems, possibly under the influence of the established (wrong) paradigm that the Harnack inequality is one of the driving forces of the theory of nonlinear partial differential equations, the emergence of such *metasolutions* in the context of population dynamics might be slightly shocking, as large solutions and metasolutions provide us with uncontestable evidence that the Harnack inequality is a technical device of a linear nature of doubtful interest in analyzing global nonlinear problems, as will become apparent in Section 4.9.

This might possibly explain the reaction of an anonymous reviewer of [195] who noted that a series of classical solutions and metasolutions were computed in the disc of radius 1 centered at the origin, B_1, with the choices

$$\begin{aligned} \Omega &= B_1(0) = \{x \in \mathbb{R}^2 \ : \ |x| < 1\}, \\ \Omega_{0,1} &= A(0.5, 1) = \{x \in \mathbb{R}^2 \ : \ 0.5 < |x| < 1\}, \\ \Omega_{0,2} &= B_{0.3}(0) = \{x \in \mathbb{R}^2 \ : \ |x| < 0.3\}, \\ \Omega_- &= A(0.3, 0.5) = \{x \in \mathbb{R}^2 \ : \ 0.3 < |x| < 0.5\}, \end{aligned}$$

by using pseudo-spectral methods.

What the heck is a "metasolution"? Please provide a formal definition. Okay, one is provided in (4.8), but "metasolutions" is used in the abstract and intro; definition needs to be earlier. The definition puzzles me. The "large" solution would seem to be very difficult to compute because of the singularity on the boundary. And what is the use or point of a solution that is infinite everywhere on another subdomain? Metasolutions are wierd...

I am alarmed by the references to "blow up"and "approach infinity on the boundary". Spectral methods are notoriously sensitive to singularities of the solution including singularities on the boundaries...

The serious problem with the paper is that the discontinuities of slope in coefficients of partial differential equation and the infinities on the boundary both makes the solution of partial differential equation singular within the domain.

I hate their $B_r(0)$, $A(R_0, R_1)$ notation for what are simply the disk of radius R and the annulus bounded in radius by R_0 and R_1. For goodness' sake, use conventional notation and wording: "disk of radius R, $r \in [0, R]$,"...

I am further bothered that their coefficient function $a(x)$ is nonzero only for $r \in [0.3, 0.5]$ for a problem in the unit disk. The PDE thus has a coefficient with a slope discontinuity. The function $u(r, \theta)$ will be singular on the lines $r = 0.3$ and $r = 0.5$. The usual spectral strategy would be to split the domain into three and solve the linear Helmholtz equation on $r \in [0, 0.3]$ and $r \in [0.5, 1]$, the nonlinear PDE on $[0.3, 0.5]$ and carefully match the pieces taking account of the singularities. Instead the authors blithely ignored the singularities entirely...

Although the strategy proposed by the reviewer in the previous paragraphs is the most natural one when dealing with linear problems where the Harnack inequality applies, it is of no help in dealing with *singular boundary value problems* such as those treated in [195] and in this book. Contrary to what happens in most 'academic problems,' real problems might be highly nonlinear and hence can develop internal interfaces whose numerical treatment is a top level challenge.

It is the hope of the author that the readers of this book will not be 'alarmed' by the large solutions and the metasolutions as much as the reviewer of [195] was. Although, at first glance, metasolutions might be slightly hard to digest because of the number of technicalities involved in their study, during the last two decades they have proven to be categorical imperatives to describe the dynamics of wide classes of parabolic equations and systems in the presence of spatial heterogeneities.

It should be noted that (P.5) does not fully characterize the asymptotic behavior of the solutions of (P.1) unless,

$$L_\lambda^{\min} = L_\lambda^{\max}.$$

Consequently, to characterize the exact asymptotic profiles as $t \uparrow \infty$ of the solutions of (P.1), one must face the problem of the uniqueness of the solutions to the singular problem (P.6) and some other closely related singular problems that the reader will find in Chapter 4. This is the main bulk of Part II, consisting of Chapters 6, 7 and 8, where a series of very sharp optimal uniqueness results found by the author and his coworkers will be analyzed in a self-contained way.

Finally, the main goal of Part III, formed by the last two chapters, is to reinforce the evidence that metasolutions also are categorical imperatives to describe the dynamics of huge classes of spatially heterogeneous semilinear parabolic problems. Precisely, Chapter 9 analyzes (P.1) in the more general case when $a(x)$ changes sign, giving a rather complete account of some of the most relevant recent advances in the theory of *superlinear indefinite problems*, and Chapter 10 studies a paradigmatic competing species model with a protected zone for one of the species to illustrate how large solutions and metasolutions play a pivotal role in describing the dynamics of spatially heterogeneous systems.

This book grew from the monograph [160] and the lecture notes of the Metasolutions course delivered by the author at the National Center for Theoretical Sciences, Tsing Hua University, Hsinchu (Taiwan), during July and August of 2009. The author is delighted to thank Professor Sze-Bi Hsu for his kind invitation to deliver it, as well as for his brilliant questions and sharp comments during these lectures. The time spent in Taiwan by the author was certainly unforgettable, both personally and professionally.

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Madrid

J. López-Gómez

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