Essentials of Calculus (Volume Ⅱ)

高等数学（Ⅱ）

侯书会　刘白羽　编

科学出版社

Essentials of Calculus (Volume II)

高等数学（II）

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SCIENCE PRESS
Beijing, China

内 容 简 介

本书分上、下两册出版. 上册共七章, 着重介绍一元微积分学的基础理论知识. 内容包括函数、极限、函数连续性, 导数、微分及其应用, 不定积分、定积分及其应用; 下册共六章, 着重介绍多元微分学的基础理论知识. 内容包括无穷级数、向量代数与空间解析几何, 多元函数、极限及其连续性, 多元函数的微分及应用, 重积分、曲线积分、曲面积分及常微分方程.

本书是基于多年教学经验, 兼顾国内工科类本科数学基础要求和海外学习的双重需要编写而成的. 与经典的中文微积分教材相比, 本书适当降低了难度, 突出了微积分学和后续应用型课程中常用的计算和证明方法. 在保证教材内容符合学科要求且不低于本科阶段微积分课程教学标准的前提下, 力求语言精准、简练, 以适应我国学生的外语水平和学习特点.

本书适于作为工科院校的国际班、双语教学班的高等数学教材和参考书.

图书在版编目(CIP)数据

高等数学(全2册)(英文版)侯书会, 刘白羽编. —北京: 科学出版社, 2016.5

ISBN 978-7-03-048045-3

Ⅰ. ①高… Ⅱ. ①侯… ②刘… Ⅲ. ①高等数学-高等学校-教材-英文 Ⅳ. ①O13

中国版本图书馆 CIP 数据核字(2016) 第 079043 号

责任编辑: 周金权 张中兴 / 责任校对: 彭 涛
责任印制: 徐晓晨 / 封面设计: 迷底书装

科学出版社 出版
北京东黄城根北街 16 号
邮政编码: 100717
http://www.sciencep.com

北京京华虎彩印刷有限公司 印刷
科学出版社发行 各地新华书店经销

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2016 年 5 月第 一 版 开本: 720×1000 B5
2016 年 5 月第一次印刷 印张: 29
字数: 760 000

定价: 89.00 元 (全 2 册)
(如有印装质量问题, 我社负责调换)

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Chapter 8

Infinite Series

In this chapter, we will introduce infinite sequences and series. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

8.1 Infinite Sequences

A **sequence** can be thought of as an ordered arrangement of real numbers, one for each positive integer.

$$\text{e.g. } 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots .$$

Mathematically, a sequence is defined as a function whose domain is a set of positive integers and whose range is a set of real numbers.

$$\text{e.g. } a_1, a_2, a_3, \cdots, a_n, \cdots .$$

To describe a pattern for each sequence, we can write a formula for the n^{th} term. For example,

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots \qquad \text{the formula is } \frac{1}{2^{n-1}},$$

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \cdots \qquad \text{the formula is } \frac{1}{n!},$$

$$\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{25}{36}, \cdots \qquad \text{the formula is } \frac{n^2}{(n+1)^2}.$$

For $a_n = \dfrac{n-1}{n}$, we write the first five terms

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots, \frac{n-1}{n}, \cdots .$$

The terms in this sequence get closer and closer to 1. See Figure 8.1. The sequence **converges** to 1.

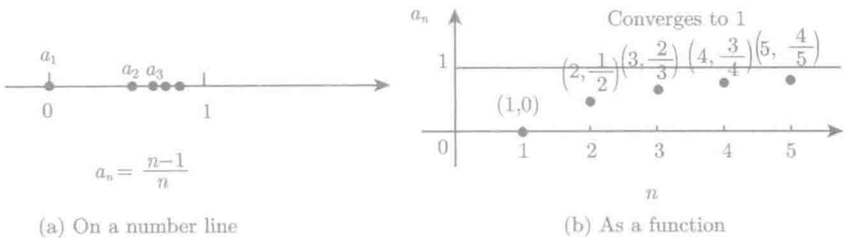

Figure 8.1 Sequence $\dfrac{n-1}{n}$

For $a_n = \dfrac{(-1)^{n+1}(n-1)}{n}$, that is

$$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \cdots, \frac{(-1)^{n+1}(n-1)}{n}, \cdots.$$

The terms in this sequence do not get close to any single value. The sequence **diverges**. See Figure 8.2.

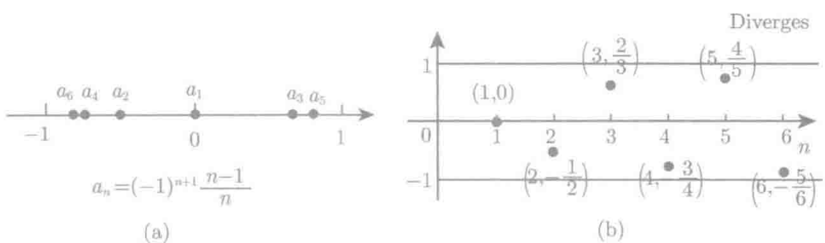

Figure 8.2 Sequence $\dfrac{(-1)^{n+1}(n-1)}{n}$

For $a_n = 3$, the terms are $3, 3, 3, \cdots, 3, \cdots$. The sequence converges to 3. See Figure 8.3.

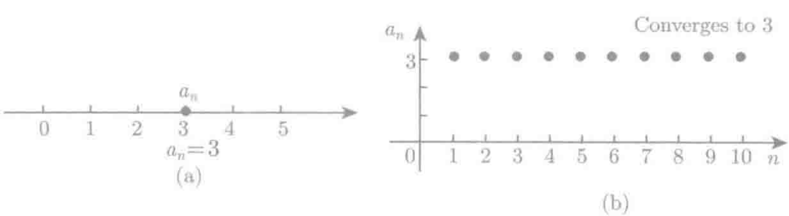

Figure 8.3 Sequence $a_n = 3$

When a sequence converges to L, $y = L$ is a horizontal asymptote. See Figure 8.4.

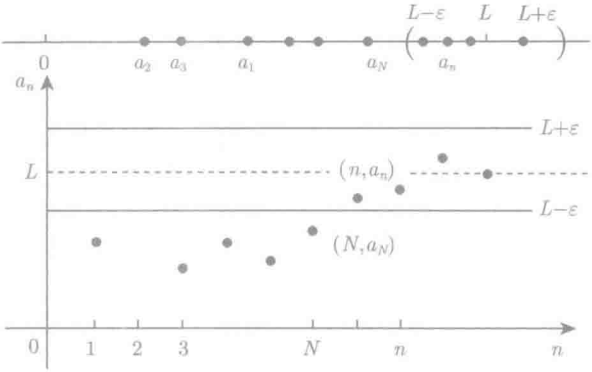

Figure 8.4 Limit as asymptote

Sequence $a_n = \dfrac{(-1)^{n+1}(n-1)}{n}$ diverges. Neither the ε-interval about 1 nor the ε-interval about -1 contains all a_n satisfying $n \geqslant N$ for some N. See Figure 8.5.

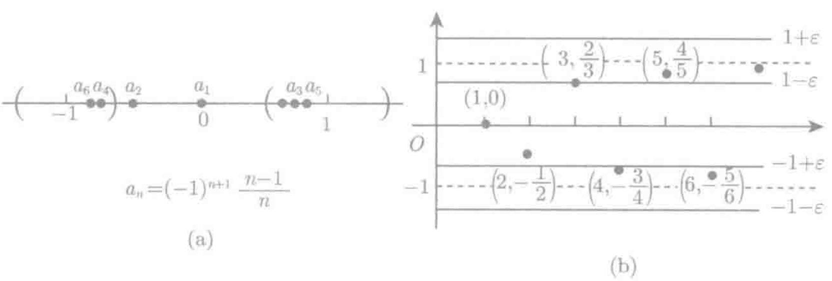

Figure 8.5 A divergent sequence

8.1.1 Convergence

Definition 8.1.1 The sequence $\{a_n\}$ is said to **converge** to L, if it has

$$\lim_{n\to\infty} a_n = L,$$

that is, if for each positive number ε there is a corresponding positive number N such that

$$n \geqslant N \Rightarrow |a_n - L| \leqslant \varepsilon.$$

A sequence that fails to converge to any number L is said to **diverge**, or to be **divergent**.

Note that (1) If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, when n is an integer, then $\lim_{n\to\infty} a_n = L$ (Converse is not true).

(2) $\lim_{n\to\infty} a_n = +\infty$ means that for every positive number M there is an integer N such that

$$a_n > M, \quad \text{whenever } n > N.$$

Example 8.1.2 $\lim_{n\to\infty} \frac{1}{n} = 0$, since if $\varepsilon > 0$ is given, then

$$\left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| < \varepsilon, \quad \text{if } n > \frac{1}{\varepsilon} = n_\varepsilon.$$

All the familiar limit theorems hold for convergent sequences. We state them without proof.

Theorem 8.1.3 (Properties of Limits of Sequences) Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences and k is a constant, then

$$\lim_{n\to\infty} k = k;$$

$$\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n;$$

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n;$$

$$\lim_{n\to\infty} a_n \cdot b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n;$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}, \quad \text{if } \lim_{n\to\infty} b_n \neq 0.$$

Observe that the limits $\lim_{n\to\infty} x_n y_n$ and $\lim_{n\to\infty} (x_n + y_n)$ may exist and be finite even if the limits $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ do not exist.

Example 8.1.4 Let $x_n = (-1)^n n$ and $y_n = \frac{1}{n^2}$. Then $\lim_{n\to\infty} y_n = 0$ and the limit $\lim_{n\to\infty} x_n$ does not exist. However, $\lim_{n\to\infty} x_n y_n = 0$.

Example 8.1.5 Let $x_n = (-1)^n$ and $y_n = (-1)^n$. Then $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} x_n$ do not exist. However, $\lim_{n\to\infty} x_n y_n = 1$.

Theorem 8.1.6 (Squeeze Theorem) Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, and $a_n \leqslant b_n \leqslant c_n$, for $n \geqslant K$ (K is a fixed integer), then

$$\lim_{n\to\infty} b_n = L.$$

Theorem 8.1.7 If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Proof Since $-|a_n| \leqslant a_n \leqslant |a_n|$, the result follows from the Squeeze Theorem. □

Theorem 8.1.8 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r. Moreover,

$$\begin{aligned} &\lim_{n\to\infty} r^n = 0, \quad \text{if } -1 < r < 1; \\ &\lim_{n\to\infty} r^n = 1, \quad \text{if } r = 1. \end{aligned}$$

Proof If $r = 0$ or $r = 1$, the result is trivial.

If $|r| < 1$, then $1/|r| > 1$, and so $1/|r| = 1 + p$ for some $p > 0$. By the Binomial Formula,

$$\frac{1}{|r|^n} = (1+p)^n = 1 + pn + (\text{positive terms}) \geqslant pn.$$

Thus,

$$0 \leqslant |r|^n \leqslant \frac{1}{pn}.$$

Since $\lim_{n\to\infty}(1/pn) = 0$, it follows from the Squeeze Theorem that $\lim_{n\to\infty} |r|^n = 0$. So $\lim_{n\to\infty} r^n = 0$.

If $|r| > 1$, then $\lim_{n\to\infty} |r^n| = \infty$. So the sequence $\{r^n\}$ diverges. □

Example 8.1.9 Compute $\lim_{n\to\infty} \dfrac{n!}{n^n}$.

Solution This is difficult to compute using the standard methods because $n!$ is defined only if n is a natural number. So the values of the sequence in question are not given by an elementary function to which we could apply tricks like L' Hospital's Rule.

Observe that $0 < \dfrac{n!}{n^n}$, for all $n > 0$. Next observe that

$$\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \cdots (n-1)n}{n \cdot n \cdot n \cdots n \cdot n} = \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n-1}{n} \cdot \frac{n}{n} < \frac{1}{n},$$

where we use $k/n < 1$. Hence

$$0 < \frac{n!}{n^n} < \frac{1}{n}.$$

Since

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

we obtain

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0$$

by the Squeeze Theorem. □

Example 8.1.10 Does the sequence

$$\frac{\sin n}{n+\cos n}$$

converge?

Solution We have $-1 \leqslant \sin n \leqslant 1$, and $-1 \leqslant \cos n \leqslant 1$, for all $n = 1, 2, 3, \cdots$. Hence, for $n \geqslant 2$

$$-\frac{1}{n-1} \leqslant \frac{\sin n}{n+\cos n} \leqslant \frac{1}{n-1}.$$

Since

$$\lim_{n\to\infty} \frac{1}{n-1} = \lim_{n\to\infty} \left[-\frac{1}{n-1}\right] = 0,$$

we conclude that the sequence $\dfrac{\sin n}{n+\cos n}$ converges and that

$$\lim_{n\to\infty} \frac{\sin n}{n+\cos n} = 0.$$ □

8.1.2 Monotonic Sequences

Definition 8.1.11 (Increasing & Decreasing Sequences) (1) A sequence $\{a_n\}$ is said to be **increasing** if $a_n \leqslant a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots$; it is said to be **decreasing** if $a_n \geqslant a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots$; a sequence $\{a_n\}$ is said to be **monotonic** if it is either increasing or decreasing.

(2) A sequence $\{a_n\}$ is **bounded above** if there is a number M such that $a_n \leqslant M$ for all $n \geqslant 1$; it is **bounded below** if there is a number m such that $m \leqslant a_n$ for all $n \geqslant 1$; a sequence $\{a_n\}$ is a **bounded** sequence if it is bounded above and below.

Theorem 8.1.12 (Monotonic Sequence Theorem) Every bounded monotonic sequence is convergent.

Proposition 8.1.13 If a sequence $\{a_n\}$ is **eventually increasing** (or **decreasing**), then there are two possibilities:

(1) There is a constant M (or m), called an **upper bound** (or **lower bound**) for the sequence, such that $L \leqslant M$ (or $L \geqslant m$). For all n, in which case the sequence converges to a limit L satisfying

$$a_n \leqslant M \quad (\text{or } a_n \geqslant m).$$

(2) No upper (or lower) bound exists, that is,

$$\lim_{n\to+\infty} a_n = +\infty \quad (\lim_{n\to+\infty} a_n = -\infty).$$

Theorem 8.1.14 (Completeness Axiom) If a nonempty set S of real numbers has an upper bound, then it has the smallest upper bound (the least upper bound), and if a nonempty set S of real numbers has a lower bound, then it has the largest lower bound (the greatest lower bound).

8.1.3 Problems for Section 8.1

1. An explicit formula for a_n is given. Write the first five terms of $\{a_n\}$ and determine whether the sequence converges or diverges. Find $\lim\limits_{n\to\infty} a_n$ if $\{a_n\}$ converges.

(1) $a_n = (-1)^n \dfrac{n}{n+2}$;

(2) $a_n = \dfrac{(-\pi)^n}{5^n}$.

2. Find an explicit formula for each sequence and determine whether the sequence converges or diverges.

(1) $\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \cdots$;

(2) $2, 1, \dfrac{2^3}{3^2}, \dfrac{2^4}{4^2}, \dfrac{2^5}{5^2}, \cdots$.

3. Write the first four terms of the sequence $\{a_n\}$. Show that the sequence converges.

(1) $a_n = \left(1-\dfrac{1}{4}\right)\left(1-\dfrac{1}{9}\right)\cdots\left(1-\dfrac{1}{n^2}\right)$, $n \geqslant 2$;

(2) $a_1 = 1$, $a_{n+1} = 1 + \dfrac{1}{2}a_n$.

8.2 Infinite Series

An **infinite series** is the sum of all the terms of a sequence:

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots .$$

Directly ("brute force") computing this sum is not possible. There is no guarantee the sum even exists! However, attempting to directly compute the sum produces a new sequence, the **sequence of partial sums**

$$S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + a_2 + \cdots + a_N.$$

Definition 8.2.1 If $\lim_{N\to\infty} S_N$ exists, we say that the infinite series **converges**, i.e. the sum can actually be computed and

$$\sum_{n=0}^{\infty} a_n = \lim_{N\to\infty} S_N.$$

If $\lim_{N\to\infty} S_N$ does not exist, we say that the infinite series **diverges**, i.e. the sum cannot be computed.

Unfortunately, at this point, determining whether or not $\lim_{N\to\infty} S_N$ exists requires first finding a formula for S_N, and that can be very difficult. Sometimes it is harder than finding the antiderivative for an integral.

There is one exception, the **geometric series**

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots .$$

For $r \neq 1$

$$S_N = \sum_{n=0}^{N} r^n = 1 + r + r^2 + r^3 + \cdots + r^N = \frac{1 - r^{N+1}}{1 - r},$$

it immediately implies

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \dfrac{1}{1-r}, & |r| < 1, \\ \text{Diverges}, & |r| \geqslant 1. \end{cases}$$

Each term of a geometric series is obtained from the preceding number multiplied