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Asymptotics and Statistical Analysis for Ruin Probabilities in Some Dependent Risk Models

(相依风险模型中破产概率的渐近性与统计分析)

Yang Yang（杨洋）

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《博士后文库》序言

博士后制度已有一百多年的历史。世界上普遍认为，博士后研究经历不仅是博士们在取得博士学位后找到理想工作前的过渡阶段，而且也被看成是未来科学家职业生涯中必要的准备阶段。中国的博士后制度虽然起步晚，但已形成独具特色和相对独立、完善的人才培养和使用机制，成为造就高水平人才的重要途径，它已经并将继续为推进中国的科技教育事业和经济发展发挥越来越重要的作用。

中国博士后制度实施之初，国家就设立了博士后科学基金，专门资助博士后研究人员开展创新探索。与其他基金主要资助“项目”不同，博士后科学基金的资助目标是“人”，也就是通过评价博士后研究人员的创新能力给予基金资助。博士后科学基金针对博士后研究人员处于科研创新“黄金时期”的成长特点，通过竞争申请、独立使用基金，使博士后研究人员树立科研自信心，塑造独立科研人格。经过 30 年的发展，截至 2015 年底，博士后科学基金资助总额约 26.5 亿元人民币，资助博士后研究人员 5 万 3 千余人，约占博士后招收人数的 1/3。截至 2014 年底，在我国具有博士后经历的院士中，博士后科学基金资助获得者占 72.5%。博士后科学基金已成为激发博士后研究人员成才的一颗“金种子”。

在博士后科学基金的资助下，博士后研究人员取得了众多前沿的科研成果。将这些科研成果出版成书，既是对博士后研究人员创新能力的肯定，也可以激发在站博士后研究人员开展创新研究的热情，同时也可以使博士后科研成果在更广范围内传播，更好地为社会所利用，进一步提高博士后科学基金的资助效益。

中国博士后科学基金会从 2013 年起实施博士后优秀学术专著出版资助工作。经专家评审，评选出博士后优秀学术著作，中国博士后科学基金会资助出版费用。专著由科学出版社出版，统一命名为《博士后文库》。

资助出版工作是中国博士后科学基金会“十二五”期间进行基金资助改革的一项重要举措，虽然刚刚起步，但是我们对它寄予厚望。希望通过这项工作，使博士后研究人员的创新成果能够更好地服务于国家创新驱动发展战略，服务于创新型国家的建设，也希望更多的博士后研究人员借助这颗“金种子”迅速成长为国家需要的创新型、复合型、战略型人才。

中国博士后科学基金会理事长

Preface

Risk theory in general and ruin probabilities in particular is traditionally considered as part of insurance mathematics, and has been an active area of research from the days of Lundberg all the way up to today. In the classical risk model, the Cramér-Lundberg theorem investigated the case of light-tailed claims under independence structure, and obtained the asymptotics for ruin probabilities. However, it would not be fair not to say that the practical relevance of the area has been questioned repeatedly. One reason is the blooming of many recent *extremal events*. Within the insurance context, extremal events present themselves spectacularly whenever some catastrophes or terrorist attacks occur, in view of the September 11, 2001 attacks, the 2004 Indian Ocean Tsunami, the 2005 Hurricane Katrina, the 2008 Sichuan earthquake, the 2010 Haiti earthquake, the 2011 Japan earthquake and, in particular, the recent financial tsunami. Extremal events may clearly correspond to individual claims which by far exceed the capacity of a single insurance company. From a mathematical point of view, such extremal events lead to some *heavy-tailed* insurance claims, which are the main object in our research. The quantifiability of such claims makes the mathematical modelling more tractable. Another reason is the independence restriction among claim sizes and claim inter-arrival times in the classical risk model. Our research concerns some dependent risk models, which admit some dependence structures existing among claim sizes and claim inter-arrival times. We mainly consider two kinds of dependent risk models, one is the model with interest rate and another is with no interest rate. Some more realistic risk models are also investigated such as the general risk model, the compound renewal risk model and the discrete-time risk model with insurance and financial risks, among others.

This book is organized as the following four chapters:

Chapter 1 presents the background of our research including the risk process, the ruin probabilities, the claim-size distributions and the claim claim arrival process. Some definitions of the heavy-tailed distribution classes commonly used in this book are introduced in this chapter. We also show the classical Cramér-Lundberg estimates for the infinite-time ruin probability in the renewal risk model with independent light-tailed claims.

Chapter 2 provides some risk models with interest rate. We firstly recall the famous Veraverbeke's theorem, which describes the tail asymptotic behavior of the supremum of a random walk with independent increments. Using this important result, the infinite-time ruin probability can be estimated in the classical Cramér-Lundberg risk model with the claims having subexponential integrated tails. We further establish the generalized Veraverbeke's theorem in the case that the increments have O-subexponential integrated distributions; meanwhile, a uniform upper bound is derived for the distribution of the supremum of a random walk with independent but non-identically distributed increments, whose tail distributions are dominated by a common tail distribution with an O-subexponential integrated distribution. In Section 2, we investigate two kinds of dependent risk models and obtain some asymptotic formulas for the infinite-time ruin probabilities. One risk model considers the claim sizes as a modulated process, and the other deals with negatively upper quadrant dependent claim sizes. In the two models, the inter-arrival times are both assumed to be negatively lower quadrant dependent. In Section 3 we focus on the finite-time ruin probability, which is more practical but much harder to investigate than the infinite-time ruin probability. Our obtained result is based on a dependent renewal risk model, where the claim sizes are independent and identically distributed with strongly subexponential tails, and the claim inter-arrival times are also negatively lower quadrant dependent. We further extend this result to a uniform one for the time horizon varying in the positive half line. In the last section of this chapter, we derive a theoretical result on the supremum of a dependent random walk with subexponential increments.

Chapter 3 focuses on some risk models with no interest rate. We firstly investigate a dependent delayed renewal risk model, where the claim sizes are pairwise negative quadrant dependent r.v.s with dominatedly-varying-tailed distributions and the claim inter-arrival times are negatively lower quadrant dependent, and derive the asymptotics for the infinite-time ruin probability. We further consider a dependent general risk model, where the claim sizes are strengthened to be negatively associated but the claim arrival process can be a general counting process. Based on the obtained results, we establish a uniform formula for the finite-time and infinite-time ruin probabilities in the described dependent delayed renewal risk model. In Section 2 we make some refinements of the results obtained in Section 1. Some weaker and more verifiable dependence structures among the claim sizes and the inter-arrival times are discussed. We remark that these newly proposed dependence structures allow many common negatively dependent random variables as well as some positively dependent ones. Section 3 considers the ordinary and compound renewal risk models. The latter is a natural modification of the classical one, where the claims at each accident moment are aggregated from a number of individual claims, meanwhile, in an ordinary renewal risk model one claim at each accident time appears. And the individual claim sizes are assumed to belong to the class of subexponential distributions rather than the heavily heavy-tailed class. We also perform some simulations to verify the approximate relationships in our obtained theoritical results.

In Chapter 4, we are interested in a discrete-time risk model model with insurance and financial risks. By establishing some results on tail behavior for the finite and infinite randomly weighted sums, we derive some asymptotic estimates for the finite-time and infinite-time ruin probabilities, respectively. In this chapter, we consider two kinds of randomly weighted sums: one is the finite randomly weighted sums where there exists some dependence structure between the primary random variables (which can be regarded as the net insurance losses) and the stochastic weights (which can be interpreted as the stochastic factors); and another is the infinite randomly weighted sums where some dependence exists among the primary random variables and the stochastic weights can be arbitrarily dependent.

In summary, we have thoroughly and systematically studied some dependent heavy-tailed risk models, and mainly investigated some estimates for finite-time and infinite-time ruin probabilities.

During the preparation of the book, I benefitted a lot from critical remarks and suggestions from my colleagues and friends, and I would like to express my gratitude to Professors Lin Jinguan, Xie Fengchang, and Doctors Cao Chunzheng, Wang Kaiyong, Yan Fangrong.

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Yang Yang
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Chapter 1

Introduction

Within the insurance industry, it is not possible nor desirable to eliminate all risks. For a given situation, risk management consists in analyzing exposure to risk. Whereas much of traditional statistics studies the average and the expected, risk management has more to do with the unexpected, the rare events and extreme cases. This calls for the use of appropriate stochastic models and heavy-tail analysis.

In this book, we mainly utilize the asymptotic behavior of ruin probabilities to describe and evaluate the risk of an insurance company. To this end, we investigate many kinds of nonstandard risk models, which have two points in common: (1) some dependence structures are allowed among the claim sizes or the claim inter-arrival times; (2) the claim sizes are assumed to be heavy-tailed.

In classical risk management, typically, the mathematical constituents of the model are assumed to be independent. This is often due to the implied benefits in terms of mathematical tractability more than to the nature of the observations. In modern risk management, a central issues is the modeling of dependent risks. In fact, through many practical examples, in the last years the consciousness that the assumption of independence between the components of a risk model is not always satisfied became widely accepted. Losses resulting for car polices in the case of hail or for households due to an earthquake or a flood are definitely dependent. As already mentioned, it is natural and reasonable to consider some dependent risk models.

Why do we fucus on some heavy-tailed risk models? This is because the ruin of an insurance company is always caused by some critically huge claims, such as those in some natural or man-made disasters, rather than a lot of small-size claims. As stated in Sigma (1996), at 150 billion US dallars, the total estimated losses in 1995 amounted to ten times the cost of insured losses, an exceptionally high amount, more than half of which was accounted for by the Kobe earthquake. Natural catastrophes alone caused insured losses of 12.4 billion US dallars, more than half of which were accounted for by four single disasters costing some billion dollars each; the Kobe earthquake, hurricane "Opal" a hailstorm in Texas and winter storms combined with floods in Northern Europe. Natural catastrophes also claimed 20000 of the 28000 fatalities in the year of the report.

We call such events as extremal events. Whatever definition one takess most will agree that Table 1.1 taken from Sigma (1996) contains extremal events. When looked upon as single events, each of them exhibits some common features. They are difficult to predict a long time ahead. It should be noted that 28 of the insurance losses reported in Table 1.1 are due to natural events and only 2 are caused by man-made disasters. If looked at within the larger context of all insurance claims, they are rare events.

Extremal events in insurance and finance have (from a mathematical point of view) the advantage that they are mostly quantifiable in units of money. Extremal events are relevant especially nowadays, in view of the September 11, 2001 attacks, the 2004 Indian Ocean Tsunami, the 2005 Hurricane Katrina, the 2008 Sichuan earthquake, the 2010 Haiti earthquake, the 2011 Japan earthquake, and, in particular, the recent financial tsunami. However most such events have a non-quantifiable component which more and more economists are trying to take into account. Going back to the data presented in Table 1.1 extremal events may clearly correspond to individual (or indeed grouped) claims which by far exceed the capacity of a single insurance company; the insurance world's reaction to this problem is the creation of a reinsurance market. One does not however have to go to this grand scale. Even looking at standard claim data within a given company one is typically confronted with statements like "In this portfolio, 20% of the claims are responsible for more than 80% of the total portfolio claim amount".

By stating above that the quantifiability of insurance claims in monetary units makes the mathematical modelling more tractable, we notice that, without exception, all such extremal evens produced one or a series of huge claims, which may exceed the solvency of an insurance company and decide whether it does ruin or not.

Table 1.1 The 30 most costly insurance losses 1970–1995. Losses are in million $US at 1992 prices. For a precise definition of the notion of catastrophic claim in this context see Sigma (1996)

Losses	Date	Evens	Country
16000	08/24/92	Hurricane "Andrew"	USA
11838	01/17/94	Northridge earthquake in California	USA
5724	09/27/91	Tornado "Mireille"	Japan
4931	01/25/90	Winterstorm "Daria"	Europe
4749	09/15/89	Hurricane "Hugo"	P. Rico
4528	10/17/89	Loma Prieta earthquake	USA
3427	02/26/90	Winter storm "Vivian"	Europe
2373	07/06/88	Explosion on "Piper Alpha" offshore oil rig	UK
2282	01/17/95	Hanshin earthquake in Kobe	Japan
1938	10/04/95	Hurricane "Opal"	USA
1700	03/10/93	Blizzard over eastern coast	USA
1600	09/11/92	Hurricane "Iniki"	USA
1500	10/23/89	Explosion at Philips Petroleum	USA
1453	09/03/79	Tornado "Frederic"	USA
1422	09/18/74	Tornado "Fifi"	Honduras
1320	09/12/88	Hurricane "Gilbert"	Jamaica
1238	12/17/83	Snowstorms, frost	USA
1236	10/20/91	Forest fire which spread to urban area	USA
1224	04/02/74	Tornados in 14 states	USA
1172	08/04/70	Tornado "Celia"	USA
1168	04/25/73	Flooding caused by Mississippi in Midwest	USA
1048	05/05/95	Wind, hail and floods	USA
1005	01/02/76	Storms over northwestern Europe	Europe
950	08/17/83	Hurricane "Alicia"	USA
923	01/21/95	Storms and flooding in northern Europe	Europe
923	10/26/93	Forest fire which spread to urban area	USA
894	02/03/90	Tornado "Herta"	Europe
870	09/03/93	Typhoon "Yancy"	Japan
865	08/18/91	Hurricane "Bob"	USA
851	02/16/80	Floods in California and Arizona	USA

When we are interested in the extremal behavior of the models described above we have to ask how extremal events occur. This means we have to find appropriate mathematical methods in order to explain events that occur with relatively small probability but have a significant influence on the behavior of the whole model. A natural class of large claim distributions is given by the *subexponential distributions*. Their defining property is:

$$\lim_{x\to\infty} \frac{\mathrm{P}(X_1 + \cdots + X_n > x)}{\mathrm{P}(\max(X_1, \cdots, X_n) > x)} = 1,$$

for every $n \geqslant 2$, where $X_1, \cdots, X_n$ are independent and identically distributed (i.i.d.) nonnegative random variables (r.v.s). Thus the tails of the distribution of the sum and of the maximum of the first n claims are asymptotically of the same order. This clearly indicates the strong influence of the largest claim on the total claim amount. In insurance, heavy-tailed (subexponential) distributions are well recognized as standard models for individual claim sizes (see, e.g. Embrechts et al. (1997)). Therefore, in this book we treat some heavy-tailed risk models as our main objective.

In the rest of this chapter, we introduce the classical risk model as well as some notions and notation in risk theory, which will commonly used in this book. After a brief summary of the basic risk model in Section 1.1, we describe the distributions of the claim sizes and the claim-arrival process in Sections 1.2. In Section 1.3 we derive the classical Cramér-Lundberg estimate for ruin probabilities in the infinite horizon case based on a small claim condition.

1.1 Risk process and ruin probabilities

The basic insurance risk model goes back to the early work by Filip Lundberg (1903) who in his famous Uppsala thesis of 1903 laid the foundation of actuarial risk theory. Lundberg realized that Poisson processes lie at the heart of non-life insurance models. Via a suitable time transformation (so-called operational time) he was able to restrict his analysis to the homogeneous Poisson process. It was then left to Harald Cramér and his Stockholm School to incorporate Lundberg's ideas into the emerging theory of stochastic processes. In doing so, Cramér contributed considerably to laying the foundation of both non-life insurance mathematics as well as probability theory. The basic model, referred to in the sequel as *the Cramér-Lundberg model*, has the following structure:

Definition 1.1.1 (The Cramér-Lundberg risk model, the renewal risk model) *The Cramér-Lundberg risk model satisfies the following assumptions* (a)–(e):

(a) *The claim size process: the claim sizes* $\{X_n,\ n \geqslant 1\}$ *form a sequence of i.i.d. positive r.v.s with common non-lattice distribution* F, *finite mean* $\mu_F = \mathrm{E}X_1$ *and finite variance* $\sigma_F^2 = \mathrm{D}X_1 \leqslant \infty$.

(b) *The claim times: the claims occur at the random instants of time*

$$0 < \sigma_1 < \sigma_2 < \cdots \ \text{a.s.}$$

(c) *The claim arrival process: the claim arrival process*

$$N(t) = \sup\{n \geqslant 1 : \ \sigma_n \leqslant t\}$$

represents the number of claims within period $[0, t]$, $t \geqslant 0$, *by convention,* $\sup \varnothing = 0$.

(d) *The claim inter-arrival times*

$$\theta_1 = \sigma_1, \quad \theta_k = \sigma_k - \sigma_{k-1}, \quad k \geqslant 2, \tag{1.1}$$

are i.i.d. exponentially distributed with finite mean $\mathrm{E}\theta_1 = 1/\lambda > 0$.

(e) *The claim sizes* $\{X_n,\ n \geqslant 1\}$ *and the claim inter-arrival times* $\{\theta_n,\ n \geqslant 1\}$ *are mutually independent.*

The renewal model is given by (a)–(c), (e) *and*

(d') *The claim inter-arrival times* $\{\theta_n,\ n \geqslant 1\}$ *given in* (1.1) *are i.i.d. with finite mean* $\mathrm{E}\theta_1 = 1/\lambda > 0$.

Clearly, in the classical Cramér-Lundberg risk model, the claim arrival process $N(t)$ is a homogeneous Poisson process with intensity $\lambda > 0$. The renewal model,which was proposed by Sparre-Andersen 1957, is a slight generalization of the Cramér-Lundberg model, and is also called as the Sparre-Andersen risk model. We remark that the renewal model allows for the renewal counting process $N(t)$, which is more general than the Poisson process for the claim arrivals.

The total claim amount process $S(t)$ of the underlying portfolio is defined as $S(t) = \sum_{k=1}^{N(t)} X_k$, $t \geqslant 0$. An important quantity in the Cramér-Lundberg risk model is the total claim amount distribution (or aggregate claim-size distribution)

$$\mathrm{P}(S(t) \leqslant x) = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{n*}(x), \quad x \geqslant 0, \quad t \geqslant 0,$$

where $F^{n*}(x) = \mathrm{P}\left(\sum_{k=1}^{n} X_k \leqslant x\right)$ is the n-fold convolution of F. Define the risk process of an insurance company as

$$R(x,t) = x + ct - S(t), \tag{1.2}$$

where $x \geqslant 0$ denotes the initial capital and $c > 0$ stands for the premium income rate. The choice of c requires a net profit condition, see (1.3) below.

In the classical Cramér-Lundberg set-up, the following quantities are relevant for various insurance-related problems.

Definition 1.1.2 (Ruin) *The ruin probability in finite time (or with finite horizon) :*

$$\psi(x,T) = \mathrm{P}\left(\inf_{0\leqslant t\leqslant T} R(x,t) < 0 \middle| R(x,0) = x\right).$$

The ruin probability in infinite time (or with infinite horizon) :

$$\psi(x) = \psi(x,\infty) = \mathrm{P}\left(\inf_{t\geqslant 0} R(x,t) < 0 \middle| R(x,0) = x\right).$$

From the viewpoint of an insurance company, an obvious condition towards solvency is $c - \lambda\mu_F > 0$, implying that the risk process $R(x,t)$ has a positive drift for large t. This leads to the basic net profit condition in the renewal model:

$$\rho = \frac{c}{\lambda\mu_F} - 1 > 0. \tag{1.3}$$

The constant ρ is called the safety loading, which can be interpreted as a risk premium rate, and is the relative amount by which the premium income rate c exceeds the average amount $\lambda\mu_F$ of claim per unit time; indeed, the premium income over the period $[0,t]$ equals $ct = (1+\rho)\lambda\mu_F t$. It is sometimes stated in the theoretical literature that the typical values of the safety loading ρ are relatively small, say 10%–20%.

By definition of the risk process, ruin can occur only at the claim times σ_n, hence for $x \geqslant 0$

$$\psi(x) = \mathrm{P}\left(\sup_{n\geqslant 1} \sum_{k=1}^{n} (X_k - c\theta_k) > x\right). \tag{1.4}$$

From (1.4) it follows that, in the renewal model, the determination of the infinite-time ruin probability $\psi(x)$ is reduced to the study of the tail distribution of the