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Effective Dynamics of Stochastic Partial Differential Equations

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随机偏微分方程的有效动力学

[美] Jinqiao Duan（段金桥） Wei Wang（王伟） 著

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Dedication

To my wife, Yan Xiong, and my children, Victor and Jessica
—J. Duan
To my father, Yuliang Wang, and my mother, Lanxiu Liu
—W. Wang

Preface

Background

Mathematical models for spatial-temporal physical, chemical, and biological systems under random influences are often in the form of *stochastic partial differential equations* (SPDEs). Stochastic partial differential equations contain randomness such as fluctuating forces, uncertain parameters, random sources, and random boundary conditions. The importance of incorporating stochastic effects in the modeling of complex systems has been recognized. For example, there has been increasing interest in mathematical modeling of complex phenomena in the climate system, biophysics, condensed matter physics, materials sciences, information systems, mechanical and electrical engineering, and finance via SPDEs. The inclusion of stochastic effects in mathematical models has led to interesting new mathematical problems at the interface of dynamical systems, partial differential equations, and probability theory. Problems arising in the context of stochastic dynamical modeling have inspired challenging research topics about the interactions among uncertainty, nonlinearity, and multiple scales. They also motivate efficient numerical methods for simulating random phenomena.

Deterministic partial differential equations originated 200 years ago as mathematical models for various phenomena in engineering and science. Now stochastic partial differential equations have started to appear more frequently to describe complex phenomena under uncertainty. Systematic research on stochastic partial differential equations started in earnest in the 1990s, resulting in several books about well-posedness, stability and deviation, and invariant measure and ergodicity, including books by Rozovskii (1990), Da Prato and Zabczyk (1992, 1996), Prevot and Rockner (2007), and Chow (2007).

Topics and Motivation

However, complex systems not only are subject to uncertainty, but they also very often operate on multiple temporal or spatial scales. In this book, we focus on stochastic partial differential equations with slow and fast time scales or large and small spatial scales. We develop basic techniques, such as averaging, slow manifolds, and homogenization, to extract effective dynamics from these stochastic partial differential equations.

The motivation for extracting effective dynamics is twofold. On one hand, effective dynamics is often just what we desire. For example, the air temperature is a macroscopic consequence of the motion of a large number of air molecules. In order to decide what to wear in the morning, we do not need to know the velocity of these molecules, only their effective or collective effect, i.e., the temperature measured by a thermometer. On the other hand, multiscale dynamical systems are sometimes too complicated to analyze or too expensive to simulate all involved scales. To make progress in understanding these dynamical systems, it is desirable to concentrate on macroscopic scales and examine their effective evolution.

Audience

This book is intended as a reference for applied mathematicians and scientists (graduate students and professionals) who would like to understand effective dynamical behaviors of stochastic partial differential equations with multiple scales. It may also be used as a supplement in a course on stochastic partial differential equations. Each chapter has several exercises, with hints or solutions at the end of the book. Realizing that the readers of this book may have various backgrounds, we try to maintain a balance between mathematical precision and accessibility.

Prerequisites

The prerequisites for reading this book include basic knowledge of stochastic partial differential equations, such as the contents of the first three chapters of P. L. Chow's *Stochastic Partial Differential Equations* (2007) or the first three chapters of G. Da Prato and J. Zabczyk's *Stochastic Equations in Infinite Dimensions* (1992). To help readers quickly get up to this stage, these prerequisites are also reviewed in Chapters 3 and 4 of the present book.

Acknowledgments

An earlier version of this book was circulated as lecture notes in the first author's course *Stochastic Partial Differential Equations* at Illinois Institute of Technology over the last several years. We would like to thank the graduate students in the course for their feedback. The materials in Chapters 5, 6, and 7 are partly based on our recent research.

The first author is grateful to Ludwig Arnold for his many years of guidance and encouragement in the study of stochastic dynamical systems and stochastic partial differential equations. We have benefited from many years of productive research interactions with our collaborators and friends, especially Peter Bates, Dirk Blömker, Daomin Cao, Tomás Caraballo, Pao-Liu Chow, Igor Chueshov, Franco Flandoli, Hongjun Gao, Peter Imkeller, Peter E. Kloeden, Sergey V. Lototsky, Kening Lu, Anthony J. Roberts, Michael Röckner, Boris Rozovskii, Michael Scheutzow, Björn Schmalfuß, and Jerzy Zabczyk. The second author would especially like to thank Anthony J. Roberts, who provided him the opportunity to conduct research at the University of Adelaide, Australia. We would also like to thank our colleagues, visitors, and students at Illinois Institute of Technology (Chicago, Illinois, USA), Huazhong University of Science and Technology (Wuhan, China), and Nanjing University (Nanjing, China), particularly Guanggan Chen, Hongbo Fu, Xingye Kan, Yuhong Li, Yan Lv, and Wei Wu, for their constructive comments.

Mark R. Lytell proofread this book in its entirety. Hassan Allouba, Hakima Bessaih, Igor Cialenco, Peter E. Kloeden, and Björn Schmalfuß proofread parts of the book. Their comments and suggestions have greatly improved the presentation of this book. Finally, we would like to acknowledge the National Science Foundation for its generous support of our research.

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October 2013

Contents

1 Introduction

Examples of stochastic partial differential equations; outlines of this book

1.1 Motivation

Deterministic partial differential equations arise as mathematical models for systems in engineering and science. Bernoulli, D'Alembert, and Euler derived and solved a linear wave equation for the motion of vibrating strings in the 18th century. In the early 19th century, Fourier derived a linear heat conduction equation and solved it via a series of trigonometric functions [192, Ch. 28].

Stochastic partial differential equations (SPDEs) appeared much later. The subject has started to gain momentum since the 1970s, with early representative works such as Cabana [58], Bensoussan and Temam [33], Pardoux [248], Faris [123], Walsh [295], and Doering [99,100], among others.

Scientific and engineering systems are often subject to uncertainty or random fluctuations. Randomness may have delicate or even profound impact on the overall evolution of these systems. For example, external noise could induce phase transitions [160, Ch. 6], bifurcation [61], resonance [172, Ch. 1], or pattern formation [142, Ch. 5], [236]. The interactions between uncertainty and nonlinearity also lead to interesting dynamical systems issues. Taking stochastic effects into account is of central importance for the development of mathematical models of complex phenomena under uncertainty in engineering and science. SPDEs emerge as mathematical models for randomly influenced systems that contain randomness, such as stochastic forcing, uncertain parameters, random sources, and random boundary conditions. For general background on SPDEs, see [30,63,76,94,127,152,159,218,260,271,306]. There has been some promising new developments in understanding dynamical behaviors of SPDEs—for example, via invariant measures and ergodicity [107,117,132,153,204], amplitude equations [43], numerical analysis [174], and parameter estimation [83,163,167], among others.

In addition to uncertainty, complex systems often evolve on multiple time and/or spatial scales [116]. The corresponding SPDE models thus involve multiple scales. In this book, we focus on stochastic partial differential equations with slow and fast time scales as well as large and small spatial scales. We develop basic techniques, including averaging, slow manifold reduction, and homogenization, to extract effective dynamics as described by reduced or simplified stochastic partial differential equations.

Effective dynamics are often what we desire. Multiscale dynamical systems are often too complicated to analyze or too expensive to simulate. To make progress in

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understanding these dynamical systems, it is desirable to concentrate on significant scales, i.e., the macroscopic scales, and examine the effective evolution of these scales.

1.2 Examples of Stochastic Partial Differential Equations

In this section, we present a few examples of stochastic partial differential equations (SPDEs or stochastic PDEs) arising from applications.

Example 1.1 (Heat conduction in a rod with fluctuating thermal source). The conduction of heat in a rod, subject to a random thermal source, may be described by a stochastic heat equation [123]

$$u_t = \kappa u_{xx} + \eta(x,t), \tag{1.1}$$

where $u(x,t)$ is the temperature at position x and time t, κ is the (positive) thermal diffusivity, and $\eta(x,t)$ is a noise process.

Example 1.2 (A traffic model). A one-dimensional traffic flow may be described by a macroscopic quantity, i.e., the density. Let $R(x,t)$ be the deviation of the density from an equilibrium state at position x and time t. Then it approximately satisfies a diffusion equation with fluctuations [308]

$$R_t = K\, R_{xx} - c\, R_x + \eta(x,t), \tag{1.2}$$

where K, c are positive constants depending on the equilibrium state, and $\eta(x,t)$ is a noise process caused by environmental fluctuations.

Example 1.3 (Concentration of particles in a fluid). The concentration of particles in a fluid, $C(x,t)$, at position x and time t approximately satisfies a diffusion equation with fluctuations [322, Sec. 1.4]

$$C_t = D\, \Delta C + \eta(x,t), \tag{1.3}$$

where D is the (positive) diffusivity, Δ is the three-dimensional Laplace operator, and $\eta(x,t)$ is an environmental noise process.

Example 1.4 (Vibration of a string under random forcing). A vibrating string being struck randomly by sand particles in a dust storm [6,58] may be modeled by a stochastic wave equation

$$u_{tt} = c^2 u_{xx} + \eta(x,t), \tag{1.4}$$

where $u(x,t)$ is the string displacement at position x and time t, the positive constant c is the propagation speed of the wave, and $\eta(x,t)$ is a noise process.

Example 1.5 (A coupled system in molecular biology). Chiral symmetry breaking is an example of spontaneous symmetry breaking affecting the chiral symmetry in nature. For example, the nucleotide links of RNA (ribonucleic acid) and DNA (deoxyribonucleic acid) incorporate exclusively dextro-rotary (D) ribose and D-deoxyribose, whereas the enzymes involve only laevo-rotary (L) enantiomers of amino acids. Two continuous fields $a(x,t)$ and $b(x,t)$, related to the annihilation for L and D, respectively, are described by a system of coupled stochastic partial differential equations [158]

$$\partial_t a = D_1 \Delta a + k_1 a - k_2 ab - k_3 a^2 + \eta_1(x,t), \tag{1.5}$$
$$\partial_t b = D_2 \Delta b + k_1 b - k_2 ab - k_3 b^2 + \eta_2(x,t), \tag{1.6}$$

where x varies in a three-dimensional spatial domain; D_1, D_2(both positive) and k_1, k_2 are real parameters; and η_1 and η_2 are noise processes. When $D_1 \ll D_2$, this is a slow-fast system of SPDEs.

Example 1.6 (A continuum limit of dynamical evolution of a group of "particles"). SPDEs may arise as continuum limits of a system of stochastic ordinary differential equations (SODEs or SDEs) describing the motion of "particles" under certain constraints on system parameters [7,195,196,207,214].

In particular, a stochastic Fisher–Kolmogorov–Petrovsky–Piscunov equation emerges in this context [102]

$$\partial_t u = D u_{xx} + \gamma u(1-u) + \varepsilon\sqrt{u(1-u)}\,\eta(x,t), \tag{1.7}$$

where $u(x,t)$ is the population density for a certain species; D, γ, and ε are parameters; and η is a noise process.

Example 1.7 (Vibration of a string and conduction of heat under random boundary conditions). Vibration of a flexible string of length l, randomly excited by a boundary force, may be modeled as [57,223]

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \tag{1.8}$$
$$u(0,t) = 0, \quad u_x(l,t) = \eta(t), \tag{1.9}$$

where $u(x,t)$ is the string displacement at position x and time t, the positive constant c is the propagation speed of the wave, and $\eta(t)$ is a noise process.

Evolution of the temperature distribution in a rod of length l, with fluctuating heat source at one end and random thermal flux at the other end, may be described by the following SPDE [96]:

$$u_t = \kappa u_{xx}, \quad 0 < x < l, \tag{1.10}$$
$$u(0,t) = \eta_1(t), \quad u_x(l,t) = \eta_2(t), \tag{1.11}$$

where $u(x,t)$ is the temperature at position x and time t, κ is the (positive) thermal diffusivity, and η_1 and η_2 are noise processes.

Random boundary conditions also arise in geophysical fluid modeling [50,51,226].

In some situations, a random boundary condition may also involve the time derivative of the unknown quantity, called a *dynamical random boundary condition* [55,79,297,300]. For example, dynamic boundary conditions appear in the heat transfer model of a solid in contact with a fluid [210], in chemical reactor theory [211], and in colloid and interface chemistry [293]. Noise enters these boundary conditions as thermal agitation or molecular fluctuations on a physical boundary or on an interface.

Noise will be defined as the generalized time derivative of a Wiener process (or Brownian motion) $W(t)$ in Chapter 3.

Note that partial differential equations with random coefficients are called *random partial differential equations (or random* PDEs*)*. They are different from *stochastic partial differential equations*, which contain noises in terms of Brownian motions. This distinction will become clear in the next chapter. Random partial differential equations have also appeared in mathematical modeling of various phenomena; see [14,279,169,175,208,216,212,228,250].

1.3 Outlines for This Book

We now briefly overview the contents of this book. Chapters 5, 6 and 7 are partly based on our recent research.

1.3.1 Chapter 2: Deterministic Partial Differential Equations

We briefly present a few examples of deterministic PDEs arising as mathematical models for time-dependent phenomena in engineering and science, together with their solutions by Fourier series or Fourier transforms. Then we recall some equalities and inequalities useful for estimating solutions of both deterministic and stochastic partial differential equations.

1.3.2 Chapter 3: Stochastic Calculus in Hilbert Space

We first recall basic probability concepts and Brownian motion in Euclidean space $\mathbb{R}^n$ and in Hilbert space, and then we review Fréchet derivatives and Gâteaux derivatives as needed for Itô's formula. Finally, we discuss stochastic calculus in Hilbert space, including a version of Itô's formula that is useful for analyzing stochastic partial differential equations.

1.3.3 Chapter 4: Stochastic Partial Differential Equations

We review some basic facts about stochastic partial differential equations, including various solution concepts such as weak, strong, mild, and martingale solutions and sufficient conditions under which these solutions exist. Moreover, we briefly discuss infinite dimensional stochastic dynamical systems through a few examples.

1.3.4 Chapter 5: Stochastic Averaging Principles

We consider averaging principles for a system of stochastic partial differential equations with slow and fast time scales:

$$du^\epsilon = \left[\Delta u^\epsilon + f(u^\epsilon, v^\epsilon)\right]dt + \sigma_1\, dW_1(t), \tag{1.12}$$

$$dv^\epsilon = \frac{1}{\epsilon}\left[\Delta v^\epsilon + g(u^\epsilon, v^\epsilon)\right]dt + \frac{\sigma_2}{\sqrt{\epsilon}} dW_2(t), \tag{1.13}$$

where ϵ is a small positive parameter and W_1 and W_2 are mutually independent Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The effective dynamics for this system are shown to be described by an averaged or effective system

$$du = \left[\Delta u + \bar{f}(u)\right]dt + \sigma_1 \, dW_1(t), \tag{1.14}$$

where the averaged quantify $\bar{f}(u)$ is appropriately defined. The errors for the approximation of the original multiscale SPDE system by the effective system are quantified via normal deviation principles as well as large deviation principles.

Finally, averaging principles for partial differential equations with time-dependent, time-recurrent random coefficients (e.g., periodic, quasiperiodic, or ergodic) are also discussed.

1.3.5 Chapter 6: Slow Manifold Reduction

We first present a random center manifold reduction method for a class of stochastic evolutionary equations in a Hilbert space H:

$$du(t) = [Au(t) + F(u(t))]dt + u(t) \circ dW(t), \quad u(0) = u_0 \in H. \tag{1.15}$$

Here $\circ$ indicates the Stratonovich differential. A random center manifold is constructed as the graph of a random Lipschitz mapping $\bar{h}^s : H_c \to H_s$. Here $H = H_c \oplus H_s$. Then the effective dynamics are described by a reduced system on the random center manifold

$$du_c(t) = [A_c u_c(t) + F_c(u_c(t) + \bar{h}^s(u_c(t), \theta_t\omega))]dt + u_c(t) \circ dW(t), \tag{1.16}$$

where A_c and F_c are projections of A and F to H_c, respectively.

Then we consider random slow manifold reduction for a system of SPDEs with slow and fast time scales:

$$du^\epsilon = [Au^\epsilon + f(u^\epsilon, v^\epsilon)], \quad u^\epsilon(0) = u_0 \in H_1, \tag{1.17}$$

$$dv^\epsilon = \frac{1}{\epsilon}[Bv^\epsilon + g(u^\epsilon, v^\epsilon)]dt + \frac{1}{\sqrt{\epsilon}} dW(t), \quad v^\epsilon(0) = v_0 \in H_2, \tag{1.18}$$

with a small positive parameter ϵ and a Wiener process $W(t)$. The effective dynamics for this system are captured by a reduced system on the random slow manifold

$$d\bar{u}^\epsilon(t) = [A\bar{u}^\epsilon(t) + f(\bar{u}^\epsilon(t), \bar{h}^\epsilon(\bar{u}^\epsilon(t), \theta_t\omega) + \eta^\epsilon(\theta_t\omega))]dt, \tag{1.19}$$

where $\bar{h}^\epsilon(\cdot, \omega) : H_1 \to H_2$ is a Lipschitz mapping whose graph is the random slow manifold.

1.3.6 Chapter 7: Stochastic Homogenization

In this final chapter, we consider a microscopic heterogeneous system under random influences. The randomness enters the system at the physical boundary of small-scale obstacles (heterogeneities) as well as at the interior of the physical medium. This system is modeled by a stochastic partial differential equation defined on a domain D_ϵ

perforated with small holes (obstacles or heterogeneities) of "size" ϵ, together with random dynamical boundary conditions on the boundaries of these small holes

$$du_\epsilon(x,t) = \Big[\Delta u_\epsilon(x,t) + f(x,t,u_\epsilon,\nabla u_\epsilon)\Big]dt + g_1(x,t)dW_1(x,t), \quad in\ D_\epsilon \times (0,T), \tag{1.20}$$

$$\epsilon^2 du_\epsilon(x,t) = \Big[-\frac{\partial u_\epsilon(x,t)}{\partial \nu_\epsilon} - \epsilon b u_\epsilon(x,t)\Big]dt + \epsilon g_2(x,t)dW_2(x,t), \quad on\ \partial S_\epsilon \times (0,T), \tag{1.21}$$

with a small positive parameter ϵ, constant b, nonlinearity f, and noise intensities g_1 and g_2. Moreover, $W_1(x,t)$ and $W_2(x,t)$ are mutually independent Wiener processes, and ν_ϵ is the outward unit normal vector on the boundary of small holes.

We derive a homogenized, macroscopic model for this heterogeneous stochastic system

$$dU = \big[\vartheta^{-1}\text{div}_x\big(\bar{A}\nabla_x U\big) - b\lambda U + \vartheta \bar{f}\big]dt + \vartheta g_1\, dW_1(t) + \lambda g_2\, dW_2(t), \tag{1.22}$$

where ϑ and λ are characterized by the microscopic heterogeneities. Moreover, $\bar{A}$ and $\bar{f}$ are appropriately homogenized linear and nonlinear operators, respectively. This homogenized or effective model is a new stochastic partial differential equation defined on a unified domain D without small holes and with the usual boundary conditions only.

2 Deterministic Partial Differential Equations

Examples of partial differential equations; Fourier methods and basic analytic tools for partial differential equations

In this chapter, we first briefly present a few examples of deterministic partial differential equations (PDEs) arising as mathematical models for time-dependent phenomena in engineering and science, together with their solutions by Fourier series or Fourier transforms. Then we recall some equalities that are useful for estimating solutions of both deterministic and stochastic partial differential equations.

For elementary topics on solution methods for linear partial differential equations, see [147,239,258]. More advanced topics, such as well-posedness and solution estimates, for deterministic partial differential equations may be found in popular textbooks such as [121,176,231,264].

The basic setup and well-posedness for stochastic PDEs are discussed in Chapter 4.

2.1 Fourier Series in Hilbert Space

We recall some information about Fourier series in Hilbert space, which is related to Hilbert–Schmidt theory.

A vector space has two operations, addition and scalar multiplication, which have the usual properties we are familiar with in Euclidean space $\mathbb{R}^n$. A Hilbert space H is a vector space with a scalar product $\langle \cdot, \cdot \rangle$, with the *usual* properties we are familiar with in $\mathbb{R}^n$; see [198, p. 128] or [313, p. 40] for details. In fact, $\mathbb{R}^n$ is a vector space and also a Hilbert space.

A separable Hilbert space H has a countable orthonormal basis $\{e_n\}_{n=1}^{\infty}$, $\langle e_m, e_n \rangle = \delta_{mn}$, where δ_{mn} is the Kronecker delta function (i.e., it takes value 1 when $m = n$, and 0 otherwise). Moreover, for any $h \in H$, we have Fourier series expansion

$$h = \sum_{n=1}^{\infty} \langle h, e_n \rangle e_n. \tag{2.1}$$

In the context of solving PDEs, we choose to work in a Hilbert space with a countable orthonormal basis. Such a Hilbert space is a separable Hilbert space. This is naturally possible with the help of the Hilbert–Schmidt theorem [316, p. 232].

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The Hilbert–Schmidt theorem [316, p. 232] says that a linear compact symmetric operator A on a separable Hilbert space H has a set of eigenvectors that form a complete orthonormal basis for H. Furthermore, all the eigenvalues of A are real, each nonzero eigenvalue has finite multiplicity, and two eigenvectors that correspond to different eigenvalues are orthogonal.

This theorem applies to a strong self-adjoint elliptic differential operator B,

$$Bu = \sum_{0\le|\alpha|,|\beta|\le m} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x)D^{\beta}u), \quad x \in D \subset \mathbb{R}^n,$$

where the domain of definition of B is an appropriate dense subspace of $H = L^2(D)$, depending on the boundary condition specified for u.

When B is invertible, let $A = B^{-1}$. If B is not invertible, set $A = (B + aI)^{-1}$ for some a such that $(B+aI)^{-1}$ exists. This may be necessary in order for the operator to be invertible, i.e., no zero eigenvalue, such as in the case of the Laplace operator with zero Neumann boundary conditions. Note that A is a linear symmetric compact operator in a Hilbert space, e.g., $H = L^2(D)$, the space of square-integrable functions on D.

By the Hilbert–Schmidt theorem, eigenvectors (also called *eigenfunctions* or *eigenmodes* in this context) of A form an orthonormal basis for $H = L^2(D)$. Note that A and B share the same set of eigenfunctions. So, we can claim that the strong self-adjoint elliptic operator B's eigenfunctions form an orthonormal basis for $H = L^2(D)$.

In the case of one spatial variable, the elliptic differential operator is the so-called Sturm–Liouville operator [316, p. 245],

$$Bu = -(pu')' + qu, \quad x \in (0, l),$$

where $p(x)$, $p'(x)$ and $q(x)$ are continuous on $(0, l)$. This operator arises in solving linear (deterministic) partial differential equations by the method of separating variables. Due to the Hilbert–Schmidt theorem, eigenfunctions of the Sturm–Liouville operator form an orthonormal basis for $H = L^2(0, l)$.

2.2 Solving Linear Partial Differential Equations

We now consider a few linear partial differential equations and their solutions.

Example 2.1 (Wave equation). Consider a vibrating string of length l. The evolution of its displacement $u(x, t)$, at position x and time t, is modeled by the following wave equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \tag{2.2}$$

$$u(0, t) = u(l, t) = 0, \tag{2.3}$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \tag{2.4}$$

where c is a positive constant (wave speed), and f, g are given initial data. By separating variables, $u = X(x)T(t)$, we arrive at an eigenvalue problem for the Laplacian ∂_{xx}