HEP World's Classics

Development of Mathematics in the 19^{th} Century

Vol. I

数学在 19 世纪的发展

第 I 卷

FELIX KLEIN

TRANSLATED BY M. ACKERMAN

HIGHER EDUCATION PRESS

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This book reproduces the text of the original edition. The content and language reflect the beliefs, practices and terminology of their time, and have not been updated.

本书是 F. 克莱因的名著，内容是作者在临终前一两年给部分同事所做的讲演，由他的学生们编辑成书。书中介绍了数学科学在 19 世纪的发展。在本卷（第一卷）中，克莱因非常详尽而且有批判性地分析了高斯、黎曼、魏尔斯特拉斯、柯西、伽罗瓦等一大批最重要的数学家的数学思想和贡献；同时也介绍了一大批物理学（特别是数学物理学）大师如开尔文、麦克斯韦、亥姆霍兹的思想和业绩；详细讨论了一些最重要的数学分支（函数论、射影几何、代数几何等）的缘起和前景。

本书适合从事数学研究和教学的本科水平以上的学生和教师学习参考，也适合研究科学史、数学史和关心、研究一般的科学思想文化发展的读者阅读。

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HEP World's Classics

There is a Chinese saying: "It is beneficial to open any book." It is even more fruitful to open and read classic books. The world is keeping on changing, but really fundamental and essential things stay the same since there is nothing new under the sun. Great ideas have been discovered and re-discovered, and they should be learnt and re-learnt. Classic books are our inheritance from all the previous generations and contain the best of knowledge and wisdom of all the people before us. They are timeless and universal. We cannot travel back in time, but we can converse with the originators of current theories through reading their books. Classic books have withstood the test of time. They are reliable and contain a wealth of original ideas. More importantly, they are also books which have not finished what they wanted or hoped to say. Consequently, they contain unearthed treasures and hidden seeds of new theories, which are waiting to be discovered. As it is often said: history is today. Proper understanding of the past work of giants is necessary to carry out properly the current and future researches and to make them to be a part of the history of science and mathematics. Reading classic books is not easy, but it is rewarding. Some modern interpretations and beautiful reformulations of the classics often miss the subtle and crucial points. Reading classics is also more than only accumulating knowledge, and the reader can learn from masters on how they asked questions, how they struggled to come up with new notions and theories to overcome problems, and answers to questions. Above all, probably the best reason to open classic books is the curiosity: what did people know, how did they express and communicate them, why did they do what they did? It can simply be fun!

This series of classic books by Higher Education Press contains a selection of best classic books in natural history, mathematics, physics, chemistry, information technology, geography, etc. from the past two thousand years. They contain masterpieces by the great people such Archimedes, Newton, Lavoisier, Dalton, Gauss, Darwin, Maxwell, and hence give a panorama of science and mathematics. They have been typeset in modern fonts for easier and more enjoyable reading. To help the reader understand difficult classics better, some volumes contain introductions and commentaries by experts. Though each classic book can stand in its own, reading them together will help the reader gain a bigger perspective of science and mathematics and understand better interconnection between seemingly unrelated topics and subjects.

Higher Education Press has been the largest publisher in China. Besides the long tradition of providing high quality books for proper education and training of university and graduate students, she has also set out to provide research monographs and references books to people at all levels around the world. Higher Education Press considers it her duty to keep the world science and mathematics community informed of what has been achieved in their subjects in easy and accessible formats. This series of classic books is an integral part of this effort.

One-half of the mathematics I know comes from the book of F. Klein *Lectures on the Development of Mathematics in the 19th Century.*

Vladimir Arnold

A Note from the Publisher

Felix Klein is one of the greatest mathematicians, lecturers and expositors in the nineteenth century. He is also a mathematician with vision and emphasizes the unity of mathematics. Among his many books and papers, the book *Development of Mathematics in the 19th Century* conveys best his perspective on mathematics. No wonder that the great mathematician V.I. Arnold stated in an interview in 1996: "One-half of the mathematics I know comes from the book of F. Klein *Lectures on the Development of Mathematics in the 19th Century.*"

We believe that this book is still valuable to all people who are interested either in mathematics or in the history of mathematics. The first volume of this book was translated into English and published by Math Science Press in 1979. Unfortunately, this edition has been long out of print, and some old copies of this book are being sold on Amazon at a huge price. Some university libraries have also lost their copies. Given the value of this book, it seems important to latex and republish it at a nominal price as a service to the world mathematics community.

Higher Education Press has tried her best to obtain permission of Math Science Press for reprinting this book. Unfortunately, its owner Robert Hermann has passed away and Math Science Press has ceased to exist. HEP has not been able to find descendants of Dr. Hermann or any successor of the Press. Dr. Hermann was an established and idealistic mathematician. Motivated by his dream of unity within mathematics and the intimate connection between mathematics and physics, he started Math Science Press to write and publish many books and also commentaries on some classic books and papers with his own unique perspective. We hope that republishing this classical book by Klein fits the dream of Dr. Hermann, and he would be happy to see this newly typeset version if he were still alive. HEP will be grateful to anyone who could put it in touch with any descendant of Dr. Hermann.

Recommendation from a reader

This is a unique history book on mathematics, and everyone should read it, since it was written by one of the most active participants during the period covered by the book, who also happened to be a great expositor.

We all have heard how great Gauss is. To understand why, one of the best ways is to read the first chapter of this book. To understand how complex analysis developed, this is also the book. Basically, if one wants to know what are the major results and theories in the 19th century of mathematics, this is the book to turn to.

I would like to recommend one less known story: the competition between Klein and Poincaré when Klein was at the height of his career and Poincaré just entered the mathematics stage. The outcome of this competition was that Klein totally collapsed and his research career was over, while Poincaré established himself as the indisputable leading mathematician in the world. What did they compete for? How did they compete? The best answers to these questions can be found in the last chapter of this book.

It is perhaps inspiring and sobering to quote from a letter from Klein to Poincaré on April 3, 1882, near the peak of their competition:

> I should add that on my part I have no intention of prolonging our *terminological* disagreement (once I have added the above-mentioned footnote to your explanation). However, if I should be led to intervene in the matter anew then I would, it is true, give a very complete and frank account of it. Let us rather compete to see which of us is best equipped to advance the theory in question! ....
>
> Lively contact with mathematicians aspiring to similar ends is always for me a prior condition of my own mathematical production.

There is no other book like this in mathematics, and probably there will be no other book like this either, since there will be no mathematician in the future who has a solid command and global vision of all mathematics like Klein and writes as well as him. When the versatile mathematician V.I. Arnold said that half of his mathematics came from this book of Klein, it reminded me a Chinese saying: "Half of The Analects of Confucius rule the world." This book will make the reader wiser, and it is also fun to read.

Lizhen Ji, University of Michgan

March 24, 2016.

FELIX KLEIN (1849—1925)

FELIX KLEIN(1849—1925)

19世纪后半叶至20世纪初最重要的数学家之一。他的贡献最为人所知的可能是关于几何学的Erlangen纲领，但是实际上远不止于此，而是贯穿了几何、代数、复分析、群论和数学物理等多个方面, 例如三维拓扑中的克莱因群。他一直主张纯粹数学与应用数学的统一，数学与物理、力学的统一，在数学内部则主张各个分支的统一。他认为自己最大的贡献正是在复分析、代数与几何的统一上所做出的努力。在方法论上，他的主张逻辑思维与几何直觉的统一也是非常突出的。在他的后半生，因为健康关系不能再继续独创性的科研工作时，他又成为著名的组织者和讲解者。可以说在他一手策划和精心组织下， 哥廷根大学成为了当时最高水平的世界数学中心，为人公认地继续和发展了高斯和黎曼的光辉传统。希尔伯特就是由他延揽到哥廷根来的。他的许多著作至今仍被世界各地的人们阅读着。此后，他又以很大的精力关心数学教育的发展，例如高中学生必须懂得微积分就是他一百年前所倡导的，他认为非如此就不可能接受当代科学的成就，这一点在当今21世纪开始之时已经成了全世界数学教育界的共识。特别是在教学中贯彻数学的历史发展与当前的教学的统一，以及逻辑思维与几何直觉方面，更是十分突出。

LIE GROUPS: HISTORY, FRONTIERS

AND APPLICATIONS

VOLUME IX

DEVELOPMENT OF MATHEMATICS IN THE 19TH CENTURY

BY FELIX KLEIN

TRANSLATED BY M. ACKERMAN

APPENDICES, "KLEINIAN MATHEMATICS FROM
AN ADVANCED STANDPOINT"

BY R. HERMANN

MATH SCI PRESS

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The title page of the original version

Preface

by Robert Hermann

This is one of the few classics in the history of science which is lively, interesting, and readable; as a bonus, it was written by one of the greatest mathematical expositors and generalists, and contains sketches of mathematical ideas that were models of their kind and still very useful.

As if this were not enough motivation for translation, it is also very valuable for the light it throws on two aspects of 19th century science which are now almost completely lost to us: geometry (both differential and algebraic) and mathematical physics. We are so used to thinking in terms of the "progress" of science that it is hard for us to understand that certain matters were understood better one hundred years ago! Certainly everything in our system of graduate education is designed to brainwash students along lines that Henry Ford (History is bunk) would have heartily approved. We also have glimpses in Klein's book of that lost paradise when there was still some communication and interrelation between pure and applied mathematics, and between mathematics and physics; although the tendencies which are so marked today toward specialization and "inventing the wheel" were already begun in Klein's day. Notice that many of Klein's complaints about the intellectual atmosphere of his day have a contemporary sound!

As for the Hilbert papers (Vol. 8), I will not present comments which directly bear on the text (as I did in Vols. 1–3), but will append certain modern topics which I believe help us to understand what Klein means. However, these comments are not for beginners, but for someone who knows at least the rudiments of manifold theory. Our modern theory of differential and integral calculus on manifolds is very much the continuation of the tradition exemplified in the work of Klein and his peers. I am sure that Klein would be also very happy to see that there is some movement toward renewed contact between geometric mathematics, physics, and technology, e.g., the differential geometric approach to mechanics, the use of differential-geometric methods in nonlinear wave and field theories, and control-system theory.

One of my favorite stories of mathematical mythology is that Klein and Lie divided up group theory between them–Klein took discrete groups and Lie took the continuous ones. (I have forgotten where I read this, probably E.T. Bell.) In our day, there has been a vast expansion of our knowledge about discrete groups and their relation to such topics as number theory and algebraic geometry. This too is very much in the Kleinian tradition; however, since my knowledge of these areas is only tangential to my own interests, I have not attempted any comments in this direction.

Part of the flavor of the book is Klein's outspoken views on science, culture, and individuals. Some are profound and significant, while others are petty, mean and bigoted. However, I do feel there is much that is of value in his arguments for the importance of intuition in mathematics, and the role that applications can play in guiding this intuitive development. We have had fifty years now of "pure" mathematics, developing only in response to its own inner logic, and I would argue that the results so far are inferior to the 19th century glories that Klein recounts here. Indeed, perhaps what is best in today's mathematics is the continuation of what was begun in the 19th century!

I would like to thank Michael Ackerman for his Herculean labors in these translations. Those who have read the manuscript have commented how fluently and elegantly he has put into English Klein's complicated German prose. Thanks also to Karin Young for her fine typing.

Contents

Introduction

[1]

There are many attempts at comprehending the intellectual life of our times and at presenting its most important aspects in compact and surveyable form. It is plain to anyone interested in mathematics that our science must not be missing from any such survey of contemporary cultural factors. One must try to give mathematics the position which is its due as one of the oldest and noblest activities of the human spirit and one of the directing forces of its development–a position which, unfortunately, it seldom receives in the minds of the educated, at least in Germany. One circumstance bears the main responsibility for this unhappy situation, and it continues to place the greatest difficulties in the path of any resolution. Unlike any other science, mathematics is built on a few principles in accordance with inescapable laws. Its exclusive character, which distinguishes its development from that of any other product of the mind and confers on it its famous "clarity", at the same time renders it the least accessible of all sciences. For whoever would penetrate it must, step by step, repeat its whole development within himself, by his own labor. Thus it is impossible to grasp even *one* mathematical concept without having assimilated all the concepts which led up to its creation, and their connections.

This sharp isolation of mathematics naturally makes it quite unsuitable for the layman's interest. For his goal is only to grasp approximately the essence of a subject which is strange to him, and to divine something of its particularity and beauty. If, nevertheless, something is to be achieved toward this end, it must come under a strong restriction on what is really desirable. It can only be a matter of giving a picture of what mathematicians try to do, of the boundlessness of the problems which our science, in its constant advance, tries to incorporate into its domain. And, I would say, without a certain "pious fraud" it cannot be done. Everything systematic, whose understanding would require special work, must be kept to a minimum. On the other hand, the historical development must be placed in the foreground. For the reader will be drawn involuntarily by his natural interest in the development of a thing. He will believe that he has come closer to it, while, in reality, he has grasped only its outer form. This is that "pious fraud" without which no popular presen-

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The page number in square brackets in the margin of the book is the page number of the original English version.

tation of this closed domain can manage. Finally, an emphasis on the influence of mathematics on neighboring fields and a vivid description of its relations to our total cultural life will offer something of a starting point for every educated reader.

Describing the development of mathematics in the 19th century meets with considerably greater difficulties than is the case with Antiquity or the Middle Ages, or
[2] even the 16th, 17th, and 18th centuries. For a history of the mathematics of Antiquity and the Middle Ages needs to treat only the relatively elementary topics; and the 16th, 17th and 18th centuries form an epoch with an essentially unified character, and its results can easily be grasped by outsiders through its relations with the fields neighboring mathematics. But a comparison with the 19th century shows us at once how different the circumstances are here.

The most important part of that earlier epoch was the development of differential and integral calculus, starting around 1700, which provided totally new possibilities for mastering mechanics and astronomy. It reached its high point in two works by French mathematicians, which, though completed in the 19th century, belong to the 18th in form and content. They are:

Lagrange, *Méchanique analytique* [Analytical Mechanics] 2 vols. 1811–1815 (first edition in one volume, 1788).

Laplace, *Méchanique céleste* [Celestial Mechanics] 5 vols. 1799–1825.

Any mathematician who is interested in the development of his science must, even today, know these works. Perhaps I should also mention:

Legendre, *Exercices de calcul intégral* [Exercises in Integral Calculus] 3 vols. 1811–1819,

because the results, up to that time, of investigating the integral are compiled here and worked out with an emphasis on numerical computation. (Tables of elliptic and Eulerian integrals; recall that decimal fractions became established from 1500 on and were followed by the discovery of logarithms around 1600.)

Beside these great achievements of applied mathematics, there was no lack of similar advances in pure mathematics during the 18th century. I might mention Newton's work *Enumeratio linearum tertii ordinis* [Enumeration of Lines of the Third Order], and Euler and Lagrange, to whom we owe great advances in algebraic equations, a great deal of number theory, and the addition theorem for elliptic integrals, to mention only a few. On the whole, however, independent works of pure mathematics were overshadowed by the powerful creations in which pure and applied mathematics united to answer the demands of the times.

The 19th century shows a totally different character, of course applied mathematics did not come to a stop. Rather, it took on larger and larger new territories. For evidence of this it suffices to recall the creation of the whole of "mathematical physics", our theoretical tool in all areas of physics except mechanics. But now pure mathematics came forward mightily in two, equally significant, ways, whole new areas were created, such as the theory of functions of a complex variable and projective geometry; secondly, the inherited possessions of science were subjected to critical examination in answer to the reawakened feeling for rigor, which had
[3] been somewhat repressed by the 18th century, in its surfeit of new discoveries.