HEP World's Classics

AN INTRODUCTION TO HIGHER MATHEMATICS

VOLUME II

高等数学引论

第二卷

HUA LOO-KENG

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Translated by Peter Shiu

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Preface

The material in this book is based on my notes when I gave a course of lectures on advanced mathematics in 1963 at the University of Science and Technology of China. The main part of the course was on complex function theory, and the material was intended to be published separately as Part I of Volume Two[1] to my *Introduction to Higher Mathematics*.

At the end of the lecture course the material was thoroughly revised and much expanded, but unfortunately the manuscripts for the revised material are now irretrievably lost. The content here is thus based on the original lithographic material left at the University of Science and Technology. Considering that the first volume of *Introduction to Higher Mathematics* was published some fifteen years ago, readers must have hoped that the second volume would have been published some time ago. If this part is to be rewritten completely, the nature of my work load and the condition of my health would mean that the project would have to be further delayed. Fortunately the basic material still reflects much of the author's opinion and attitude, and in any case one cannot afford to be too fussy when faced with time constraints. I now present the material as it is in this volume, and it is my hope that, after studying the material, readers will offer me their valuable suggestions so that there will be a revised edition.

Hua Loo-Keng
Beijing Hospital, 19 January, 1978

My original intention was to write six or seven volumes to *Introduction to Higher Mathemtics*. The first volume, in two parts,[2] was published in 1963. However, the drafts for most of the rest for the project were lost, leaving only scattered bits and pieces. Nevertheless, Part One of Volume II managed to see the light of day in 1981. There was some yearning that I would return to the original plan for the project, but after the realisation that the drafts were irretrievably lost, and my assessment that time was not on my side, I reluctantly had to accept that there was no real prospect for me to rewrite the lost material. Nevertheless, I

[1] Translator's footnote: In this English edition, Volume I comprises Parts I and II of the original Volume One, published in 1962 in Chinese. Thus Chapters 1–14 in this current edition comprise what the author refers to here as 'Part I of Volume Two'.

[2] Translator's footnote: In this English edition, 'Volumes I and II' are what were Parts One and Two of Volume I in the first edition in Chinese, and Volume III in this edition is what the author referred to in this preface as his 'Part One of Volume II'.

have decided to revise whatever was left of my notes for the lecture course given in 1962 at the University of Science and Technology in China, and to publish them piecemeal (with the same, or perhaps a different, title).

Fortunately, this part of the project can be considered to be reasonably independent of the other parts. When lecturing or studying the material, the reader should bear in mind the following: The teacher ought to be aware of the level of knowledge of the students, and should not advance too fast, while the student should not feel that we are only repeating rather simple material. The main reason is that we are essentially taking the second step in the '1, 2, 3; n; ∞' approach, as in the study of matrices, for example. Thus, in the preparation volumes, we spoke of one, two and three variables in one, two and three dimensions, whereas we are now dealing with n variables in an n-dimensional space. There were twelve chapters altogether in the original draft to this volume, and I recall that differential geometry in n dimensions was discussed in the final three chapters. Although the material is now lost, nevertheless the reader may wish to develop it, using as a model what has been presented on the differential geometry of the curve in space given in the first volume. One has to use different types of matrices corresponding to the underlying orthogonal group in order to derive the differential properties associated with curves in an n-dimensional space. Actually this is an excellent exercise, and, when completed satisfactorily, one can then replace the orthogonal group by other groups and study the properties of the differential invariants.

Time not being on my side, I regret that I cannot do much revision to the manuscript, and I beg the diligent reader to make the necessary corrections.

Hua Loo-Keng
11 October 1981

I should have added that, during the year when I wrote the material at the University of Science and Technology, comrade Gong Sheng had given me a lot of assistance, and he was even more helpful later in the search for the lost material.

Gong Sheng's diligence and attitude went beyond his responsibility as a comrade, and I would like to record my gratitude to him and to thank him.

Hua Loo-Keng
9 September 1983

Translator's note

In the *Biographical Memoir* (Vol. 81, 2002) of the (American) National Academy of Sciences, Professor Halberstam wrote:

> If many Chinese mathematicians nowadays are making distinguished contributions at the frontiers of science and if mathematics in China enjoys high popularity in public esteem, that is due in large measure to the leadership Hua gave his country, as scholar and teacher, for 50 years.

In 1979 the author and Halberstam were introduced to each other, and they immediately became good friends. As gifts for each other, *Sieve Methods* from Halberstam, and *Introduction to Number Theory* from the author, were exchanged, and I was asked by both of them to have the latter translated into English. I was overwhelmed with the honour of being asked, and also with fear because my Chinese was no longer fluent, not to mention the depth of the mathematics involved. The author's only piece of advice to me was to read the whole book thoroughly first. It was sound advice because I regained my confidence in the language, and also managed to learn the mathematics. Years later Professor Wang Yuan persuaded me that I could also take on the translation of his fine biography of Teacher Hua Loo-Keng, and I learned much more about the man, of course.

In fact, Hua had mentioned to me the existence of the titles to the present volumes, indicating that he would also like these books to be translated into English one day. The invitation from Higher Education Press (China) to take on the task therefore came as no surprise. Once again I was overwhelmed by the honour, and this time the fear of my inadequacy lies in the breadth of the mathematics involved. My memory of the man, and his sense of duty, left me no doubt that I should take on the task. It only remains for me to thank HEP for their efficiency, and also Cambridge University Press for providing me with their expertise.

Peter Shiu
Sheffield, August 2011

Hua Loo-Keng and *Introduction to Higher Mathematics*

WANG YUAN
Institute of Mathematics, Chinese Academy of Sciences

I

The University of Science and Technology of China was established in September 1958. Following the policy of 'pooling the entire strength of the Chinese Academy of Sciences for the development of the University and combining the institutes of the Academy with the departments at the University', Hua Loo-Keng and other distinguished scholars from the Academy, such as Wu Youxun, Yan Jici and Qian Xuesen, took up positions and gave lectures there.

Hua Loo-Keng became Head of the Applied Mathematics Department (later renamed the Mathematics Department). He then initiated the 'single-dragon method' for mathematical instructions. Hua had always emphasised the interdependent relationships between all branches of mathematics, and considered the separation of such foundation subjects as calculus, algebra and complex function theory into compartments for teaching to be artificial. He therefore decided to put all such foundation materials together for a single course, which he taught for some three to four years.

Frankly speaking, the production of teaching materials for the 'single-dragon method' could only be undertaken by a mathematician who had a complete command of the whole subject together with a true understanding of the relationships between the various branches. On the other hand, there is no doubt that Hua Loo-Keng was admirably suitable for such an undertaking. This is because Hua himself had made significant contributions in several areas of mathematics and he well understood the subtle interrelationships across the various subjects.

We note the following special points from the four volumes that have already been published.[1]

First, for a university foundation course on mathematics, it is important to deal with the basics. Hua Loo-Keng had often said the following:

> I love to lecture in such a way that the new material seems to be already familiar, while the old material is reviewed in a new setting. It should appear that we are just introducing some new stuff

[1] The new English language edition comprises two volumes: Volume I contains the first two and Volume II the second two previously published volumes. In this Introduction all references to volume and chapter numbers refer to the original, four-volume, work.

into rather familiar territories. This is because, speaking generally, when the new is compared to the old there is not much that is really new or very different; often it is just a new line of thought with the same material being dressed up in new clothes only. If we get used to reviewing and revising in this way it becomes easy to discover and understand the genuinely innovative ideas ... The 'difference' and 'similarity' between 'numbers' and 'shapes', and the 'abstract' and the 'concrete' etc., are over and over again mere transitions of essentially the same objects, but perhaps set in a new environment.

Secondly, given a sufficient command of the mathematics, when it comes to the exposition of any topic, we can almost always consider it as a part of a more general scheme.

Thirdly, Hua Loo-Keng was a very hard-working mathematician who would never dismiss the easy material as trivial. He invested much of his time and effort practising 'shadow boxing', frequently using basic mathematics to deliver an interesting or pertinent application. The reader should feel that the mathematics here is really alive, and yet never so esoteric that it has become out of reach.

Fourthly, Hua Loo-Keng's prevailing attitude towards mathematics is to employ a 'direct method'. Without any hesitation he would use the simplest and the most elementary tools to tackle even the most difficult problems. He would never start from the abstract, and would always consider a concrete example as a first approach to the problem, before following it up with more general conclusions. In writing these books, he kept faith with this attitude. The proofs of theorems are not lengthy, most of the time never more than one or two pages, so that the reader can absorb them with ease.

II

In Volumes One and Two, the materials covered are calculus, advanced calculus and analysis. The theory of real numbers is presented in the first chapter. Hua uses the infinite decimal system to define the real numbers; although the presentation is deliberately on the descriptive side, the treatment is still quite rigorous. After this, the ϵ-N approach is given via Cauchy sequences. In the supplement to the first chapter, besides writing about the use of the binary system in computers, Hua also shows that a necessary and sufficient condition for a number to be rational is that there is periodicity in its decimal expansion. He then goes on to consider rational approximations to real numbers, and the use of 'continued fractions'. This is the usual material for 'elementary number theory', but he now applies the theory to calculations, first involving the solar and lunar calendars and then various astronomical phenomena, such as eclipses and conjunctions.

The concept of a 'limit' is often the first difficulty to be encountered by students in their transition from school to university mathematics. Here Hua goes back to talking about limits for numbers again in Chapter 4. The notions of upper and lower limits are introduced, and the concept of continuity and the ϵ-δ theory are then discussed. There will be yet more on limits, of course, but the reader will have had its gentler introduction.

The second chapter is on vector algebra, and the main material is concerned with various geometric formulae in Euclidean space; the 'supplement' includes material on spherical trigonometry and the application of the vector method to Newtonian mechanics.

After the consideration of continuity it is natural to deal with the applications of the differential and integral calculus. In Chapter 10 the Euler summation formula is introduced: *Let $\phi(x)$ have a continuous derivative in $[a, b]$. Then*

$$\sum_{a<n\leq b} \phi(n) = \int_a^b \phi(x)dx + \int_a^b \left(x - [x] - \frac{1}{2}\right)\phi'(x)dx + \left(a - [a] - \frac{1}{2}\right)\phi(a) - \left(b - [b] - \frac{1}{2}\right)\phi(b),$$

where $[x]$ denotes the integer part of x.

Hua first applies Euler's formula to deduce Stirling's formula, and then goes on to use the summation formula to derive the various approximation formulae for integrals, such as the trapezium and Simpson rules, together with estimates for the error terms. This is different from the usual approach and, apart from seeing the generality of the Euler summation formula, the reader can see immediately how the dominating term for a sum can be given the integral representation.

In Chapter 13 the author deals with sequences of functions, leading to a deeper consideration of limit. This involves the notion of uniform convergence and the associated criteria for its application to infinite products, differentiation under the integral sign, and the interchange of orders for integration and summation. Besides the various results associated with the differential and integral calculus, the chapter also includes materials on the contraction mapping principle and the theorem of Cauchy–Kovalevskaya.

In the supplement for Chapter 15 the author discusses various methods in the calculation of area and volume and their practical applications. Such methods were first found in texts on geography and mining, using elementary geometry to effect the required calculations which are usually rather complicated. Here, in only a dozen pages, the author sets out the basic ideas and develops a theory to obtain the analytic results.

Differential geometry is the subject matter in Chapters 14 and 18. Having dealt with the differential calculus, the new notion of differential geometry should no longer be that formidable. This includes the usual results in the subject such as the first and second differential forms of Gauss, curvature, tensor and the formulae of Gauss and Codazzi.

Chapter 19 is on Fourier series and covers the usual material in the subject.

Chapter 20 is on systems of ordinary differential equations. Here the author introduces the formulae associated with the orbits for artificial satellites, including the three laws governing their motions, and a discussion of the many-bodies problem. Some of the materials are taken from interesting exercises considered by the author soon after the launching of the first artificial satellite from the USSR.

III

The third volume is mainly concerned with 'complex variable theory', but much else is also included. The author starts with the geometry of the complex plane, from where the Möbius transformation is introduced, together with discussions on the linear group, the

Riemann sphere, cross-ratios, harmonic points, etc. Finally the fundamental theorem (von Staudt) of projective geometry is established:

A one-to-one continuous transformation in the (complex) projective space into itself which maps the harmonic points to themselves must be a generalised linear transformation.

This important theorem has relevance in the study of the space of matrices and is related to some of the research work by the author in matrix geometry.

The second chapter is on non-Euclidean geometry. The author introduces us to parabolic (Euclidean) geometry, spherical (elliptic) geometry and hyperbolic (Lobachevsky) geometry. Here the reader will see the definitions for the various different 'metrics'.

The third chapter is on analytic functions and harmonic functions. The author leads us to the important Riemann mapping theorem:

Let D be a simply connected region with more than one boundary point, and let z_0 *be an interior point with a certain specified direction. Then there is a unique conformal transformation mapping D bijectively onto the unit circle with* z_0 *being mapped to the origin, with the specified direction along the positive real axis.*

Hua then writes that, with this theorem, a problem in a simply connected region can now be transformed to one in the unit circle, and thus if we can understand clearly what is happening in the unit circle then there is hope for a solution to the more general situation. This is because, within the unit circle, there is now the useful tool of power series representation for a complex function. Throughout the text we often see the author emphasising such points.

Cartan has proved the following:

There are six types of irreducible homogeneous bounded symmetric domains for analytic mappings. Two of these are exceptional and the other four are the so-called classical domains.

Classical domains may be regarded as the higher dimensional analogues of the unit disc and other domains in the complex plane. Therefore the importance of the unit disc is easily recognised. Hua established the theory of harmonic analysis on classical domains (an orthogonal system for each of the four classical domains), and consequently he obtained the Cauchy kernel, the Poisson kernel, etc. for the classical domains. Hua's idea on this research lies in his deep understanding of the theory of functions on the unit disc.

By applying Riemann's theorem many important results in complex function theory become much easier to establish and understand.

In Chapter 5 the author introduces the distance function and uses it to define convergence. This then extends the notion of convergence, reflecting on what has been said about new material appearing to be familiar already.

In this volume, besides the theory of complex functions, the author also gives us plenty of other material. In Chapter 11 we have various summability methods applied to some divergent series; they include the methods of Cesàro, Hölder, Borel and Abel. Indeed there are even Tauberian theorems and other related material which is usually considered only in treatises on the theory of Fourier series.

Chapter 12 deals with problems arising from solutions to differential equations, such as the Riemann–Hilbert problems and mixed types of differential equations.

The elliptic functions of Weierstrass and Jacobi together with other more recent results in number theory are dealt with in Chapters 13 and 14; in fact it is rare that such materials are given in a university course.

IV

The main material in Volume Four is concerned with the algebra of matrices, but it goes well beyond that in scope.

Chapter 4 is on systems of difference equations and ordinary differential equations, while Chapter 5 deals with the convergence of solutions. The use of matrix theory to solve ordinary differential equations includes a discussion of Lyapunov's method.

Chapter 8 has *Volume* as the title. Here the author derives the formula for the volume of an m-dimensional manifold, and deals with the evaluation in the general case associated with an orthogonal group.

Chapter 9 is on non-negative matrices, and would have included the author's work on their applications to economics. This is the unfinished volume. The author had indicated that there would have been three more chapters on differential geometry in n-dimensional space, making use of materials on differential geometry and orthogonal groups given in the second volume.

V

The author wrote in the Prefaces to Volumes One and Two that the books were:

> *. . . written in a bit of hurry [and] done without much editing . . .*
>
> *The reader may discover that there are materials not found in other texts, or that the treatment of some material is slightly different, but not that much . . .*
>
> *Thinking in a vacuum there may be plenty of errors . . .*
>
> *When just copying down materials which are familiar, mistakes are sometimes unavoidable already. More worrying is the material that is being written for the first time. . . .*
>
> *In particular there must be places where the more advanced material has been somewhat 'dumbed-down', or a genuinely complicated argument over-simplified. We invite readers to put us back on the right path.*

Such words are, of course, a reflection of the integrity and high standard that Hua Loo-Keng set himself. In fact, much detail was given in both Volumes One and Two. However, in Volumes Three and Four, there are more than several places at which the author wrote 'Similarly it is not difficult to prove that . . .' or words to that effect. This is acceptable when writing for someone who is as knowledgeable as the author. However, when it comes to students coming across such material for the first time, or even for myself when I was working closely with him as his assistant, it was not at all easy. For this reason, when studying the material, special attention must be given to such places where one needs to put in the extra work.

There are well over a thousand pages in these four volumes. As teaching material there is obviously too much for an ordinary student and, as the author himself had pointed out, teachers need to make a judicious selection. However, for the teachers themselves, I believe the books as a whole are excellent. Indeed, capable students should be able to learn much from any of the chapters under suitable guidance from their teachers.

Apart from the rewriting of Theorem 7 in Section 3 of Chapter 4 in Volume One,[2] there had only been minor corrections of misprints during each reprinting of the books. Operating in this way, the spirit given to the original edition is naturally preserved. On the other hand, much revision could also be given, but an informed discussion is required for a decision. Perhaps this is not yet the moment.

Take the following example. The famous Bieberbach conjecture:

The coefficients of the normalised univalent function $f(z) = z + a_2z^2 + \cdots$ $|z| < 1$, *satisfy the inequality* $|a_n| \le n$, $n = 2, 3, \ldots$.

This conjecture was mentioned in Chapter 10 of Volume Three, in which the partial results of Littlewood, Nevanlinna, Dieudonné and Rogossinski were given. However, the Bieberbach conjecture was proved by L. de Branges in 1985. Whether such material should be included in a new edition is perhaps worth considering.

More to the point is the following. Hua Loo-Keng had worked tirelessly on the project: he produced six or seven manuscripts, and yet he had never spoken to any of his assistants concerning their overall plans, nor did anyone ask him about the project. Looking back at it now, it appears that material on abstract algebra, algebraic topology, Lebesgue theory and much else might also have been included in the project.

Back in the 1950s Hua spoke to me several times on the teacher and student relationship between Dirichlet and Dedekind. In the nineteenth century Dirichlet wrote a certain text (*Lectures on the Theory of Numbers*), and in each subsequent edition Dedekind wrote a 'Supplement' updating the text, so much so that the supplements eventually took up more space than the original text.

Hua Loo-Keng encouraged us to revise his texts continually, be they modifications or addition of materials. For example, there were several editions of his *Introduction to Number Theory* during his lifetime. Xiao Wenjie (P. Shiu) and myself wrote some supplements to the latest edition that Hua himself approved and much appreciated. However, for *Introduction to Higher Mathematics*, the area being covered is so enormous that I simply do not have the required knowledge and stamina to write the supplements.

Indeed, with the passage of time, those university students who attended his lectures are themselves rather old, retired and also lack the energy to take on the task. If there are to be supplements, we shall have to wait for the next generation, or perhaps even another generation, of scholars. However, I am confident that the young mathematicians in our country will keep the fire in the stove first lit by Hua Loo-Keng.

[2] See Theorem 4.21.

The year 2010 will be the centenary of the birth of Hua Loo-Keng, and also the twenty-fifth anniversary of his death. We thank Higher Education Press for the work done for the new edition of these books. At the same time they are also preparing to have an English edition for these books, which is very far sighted of them because it will be a very worthwhile undertaking.

Thinking back, it has been some fifty years since I was a student and an assistant. I was in the fortunate position to help Teacher Hua Loo-Keng with his lectures and the writing of Volumes One and Two of these books at the University of Science and Technology. The vivid memory of the time spent together will be treasured by me forever. With the new printing I take the opportunity to wish sincerely that the publication of these books will help to develop and expand mathematical knowledge in our country, and to nurture the talented ones so that new and important contributions will be forthcoming.

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