Computational Mechanics Series

Extended Finite Element Method

扩展有限单元法

Zhuo Zhuang, Zhanli Liu, Binbin Cheng, Jianhui Liao

清华大学出版社

Computational Mechanics Series

Extended Finite Element Method

扩展有限单元法

Zhuo Zhuang, Zhanli Liu,
Binbin Cheng, Jianhui Liao

清华大学出版社
北京

内 容 简 介

本书为《扩展有限单元法》英文版。扩展有限单元法是为了解决复杂断裂问题应运而生的一种新的有限元方法，对于各种形状、沿任意路径扩展的裂纹具有明显的精确性与高效性。本书较为系统地介绍了扩展有限元的理论、应用及最新研究进展，可供力学、土木、机械、航空、航天等专业的科研人员阅读参考。

图书在版编目(CIP)数据

扩展有限单元法=Extended finite element method：英文/庄茁等著. - -北京：清华大学出版社，2015
（计算力学丛书）
ISBN 978-7-302-39482-2

Ⅰ. ①扩… Ⅱ. ①庄… Ⅲ. ①有限元法—英文 Ⅳ. ①O241.82

中国版本图书馆 CIP 数据核字（2015）第 036163 号

责任编辑：石　磊
封面设计：常雪影
责任印制：宋　林

出 版 者：清华大学出版社
网　　址：http://www.tup.com.cn，http://www.wqbook.com
地　　址：北京清华大学学研大厦 A 座　邮　　编：100084
社 总 机：010-62770175　邮　　购：010- 62786544
投稿与读者服务：010-62776969，c-service@tup.tsinghua.edu.cn
质量反馈：010-62772015，zhiliang@tup.tsinghua.edu.cn
印 装 者：北京中献拓方科技发展有限公司
发 行 者：全国新华书店
开　　本：153mm×230mm　印　　张：17.75
版　　次：2015 年 5 月第 1 版　印　　次：2015 年 5 月第 1 次印刷
定　　价：79.00 元

产品编号：051674-01

Preface

The extended finite element method (X-FEM) is a novel numerical methodology, which was first proposed by Belytschko et al. in 1999. It has subsequently been developed very quickly in the mechanics field worldwide. Based on the finite element method and fracture mechanics theory, X-FEM can be applied to solve complicated discontinuity issues including fracture, interface, and damage problems with great potential for use in multi-scale computation and multi-phase coupling problems.

The fundamental concept and formula of X-FEM are introduced in this book, as well as the technical process of program implementation. The expressions for enriched shape functions of the elements are provided, which include the displacement discontinued crack, termed as "strong discontinuity", and strain discontinued interface, termed as "weak discontinuity", like heterogeneous materials with voids and inclusions, interfaces of bimaterial and two-phase flows. X-FEM can be used to simulate element-crossed cracks and element-embedded cracks. Cracks with complex geometry can be modeled by structured meshes and can propagate along arbitrary route in the elements without the need for a re-meshing process, which provides considerable savings in computation cost whilst achieving precision.

In the early 1990s, the first author of this book, Prof. Zhuo Zhuang, was working on Ph.D. research under Prof. Patraic O'Donoghue at University College Dublin, Ireland, and completed a thesis on the development of the finite element method for dynamic crack propagation in gas pipelines. In 1995, he returned to China and took an academic position at Tsinghua University. He has the privilege of learning from and working with Prof. Keh-Chih Hwang, and is striving to simulate arbitrary crack growth in three-dimensional continuities and curved shells. This is a natural choice for crack propagation, in which the original failure behavior of the structures reappears. In 2011, this aspiration was released by Dr. Binbin Cheng, who is the third author of this book. From his Ph.D. thesis work at Tsinghua University, they have developed an X-FEM code, named SAFRAC, with its own properties. The second author, Dr. Zhanli Liu, obtained a doctoral degree in 2009. The research thesis "The Investigation of Crystal Plasticity at Microscale by Discrete Dislocation and Nonlocal Theory" was nominated and achieved a national excellent doctor degree thesis in 2011. After graduation from Tsinghua University, he went to Northwestern University, USA to conduct postdoctoral research under Prof. Ted Belytschko. He continued to develop the X-FEM method for dynamic crack propagation and applications in heterogeneous materials, like ultrasonic wave propagation in three-dimensional polymer matrices enhanced by particles and short fibers. He returned to Tsinghua

University and took an academic position in 2012. The fourth author, Dr. Jianhui Liao, obtained a doctoral degree in 2011 at Tsinghua University. His thesis work focuses on the application of X-FEM simulation in two-phase flows. One professor and three former Ph.D. students are working together to write this book in order to demonstrate the research achievements on X-FEM in the last decade.

In this book, Chapter 1 reviews the development history, reference summarization, and research actuality of X-FEM. Chapters 2 and 3 provide an introduction to fracture mechanics, considering the essential concepts of static and dynamic linear elastic fracture mechanics, such as the crack propagation criterion, the calculation of stress intensity factor by interaction integral, the nodal force release technique to simulate crack propagation in conventional FEM, and so on. These two chapters provide essential knowledge of fracture mechanics essential for study of the subsequent chapters. Readers who are familiar with fracture mechanics can skip these two chapters. Chapters 4 and 5 contain the basic ideas and formulations of X-FEM. Chapter 4 focuses on the theoretical foundation, mathematical description of the enrichment shape function, discrete formulation, etc. In Chapter 5, based on the program developed by the authors and their co-workers, numerical studies of two-dimensional fracture problems are provided to demonstrate the capability and efficiency of the algorithm and the X-FEM program in applications of strong and weak discontinuity problems. In Chapters 6–9, scientific research conducted by the author's group is given as examples to introduce the applications of X-FEM. In Chapter 6, a novel theory formula and computational method of X-FEM is developed for three-dimensional (3D) continuum-based (CB) shell elements to simulate arbitrary crack growth in shells using the concept of enriched shape functions. In Chapter 7, the algorithm is discussed and a program is developed based on X-FEM for simulating subinterfacial crack growth in bimaterials. Numerical analyses of the crack growth in bimaterials provide a clear description of the effect on fracture of the interface and loading. In Chapter 8, a method for representing discontinuous material properties in a heterogeneous domain by X-FEM is applied to study ultrasonic wave propagation in polymer matrix particulate/fibrous composites. In Chapter 9, a simulation method of transient immiscible and incompressible two-phase flows is proposed, which demonstrates how to deal with multi-phase flow problems by applying X-FEM methodology. Based on the scientific research in the author's group, Chapter 10 gives the applications of X-FEM in other frontiers of mechanics, e.g. nano-mechanics, multi-scale simulations, crack branches, and so on.

This book was published in a Chinese version in 2012, and was the first book on X-FEM published in China. At that time, Dr. Zhanli Liu was a postdoctoral fellow working at Northwestern University in the USA. He presented a copy of the book to Prof. Ted Belytschko to express our respect for him. Ted was very happy to see it and made complimentary remarks about the book, although he could not follow the Chinese characters but only the equations and figures. He encouraged us to publish this book in an English version.

Regarding the English version, we would like to thank Mr. Lei Shi, Ms. Qiuling Zhang, and Ms. Hongmian Zhao at Tsinghua University Press. Without their encouragement and help, we could not have completed this book. We are also grateful to the Ph.D. candidates Ms. Dandan Xu and Mr. Qinglei Zeng for the computational examples that they provided.

This book is suitable for teachers, engineers, and graduate students on the disciplines of mechanics, civil engineering, mechanical engineering, and aerospace engineering. It can also be referenced by X-FEM program developers.

Zhuo Zhuang
Zhanli Liu
Binbin Cheng
Jianhui Liao
October 2013

Contents

Preface v

1. **Overview of Extended Finite Element Method** 1
   - 1.1 **Significance of Studying Computational Fracture Mechanics** 1
   - 1.2 **Introduction to X-FEM** 2
   - 1.3 **Research Status and Development of X-FEM** 8
     - 1.3.1 The Development of X-FEM Theory 8
     - 1.3.2 Development of 3D X-FEM 10
   - 1.4 **Organization of this Book** 11

2. **Fundamental Linear Elastic Fracture Mechanics** 13
   - 2.1 **Introduction** 13
   - 2.2 **Two-Dimensional Linear Elastic Fracture Mechanics** 15
   - 2.3 **Material Fracture Toughness** 19
   - 2.4 **Fracture Criterion of Linear Elastic Material** 20
   - 2.5 **Complex Fracture Criterion** 22
     - 2.5.1 Maximum Circumference Tension Stress Intensity Factor Theory 22
     - 2.5.2 Minimum Strain Energy Density Stress Intensity Factor Theory 24
     - 2.5.3 Maximum Energy Release Rate Theory 27
   - 2.6 **Interaction Integral** 29
   - 2.7 **Summary** 31

3. **Dynamic Crack Propagation** 33
   - 3.1 **Introduction to Dynamic Fracture Mechanics** 33
   - 3.2 **Linear Elastic Dynamic Fracture Theory** 35
     - 3.2.1 Dynamic Stress Field at Crack Tip Position 35
     - 3.2.2 Dynamic Stress Intensity Factor 37
     - 3.2.3 Dynamic Crack Propagating Condition and Velocity 38
   - 3.3 **Crack Driving Force Computation** 41
     - 3.3.1 Solution Based on Nodal Force Release 41
     - 3.3.2 Solution Based on Energy Balance 43
   - 3.4 **Crack Propagation in Steady State** 44
   - 3.5 **Engineering Applications of Dynamic Fracture Mechanics** 45
   - 3.6 **Summary** 49

4. Fundamental Concept and Formula of X-FEM **51**

4.1 X-FEM Based on the Partition of Unity **51**

4.2 Level Set Method **53**

4.3 Enriched Shape Function **55**

4.3.1 Description of a Strong Discontinuity Surface 55

4.3.2 Description of a Weak Discontinuity Surface 58

4.4 Governing Equation and Weak Form **60**

4.5 Integration on Spatial Discontinuity Field **64**

4.6 Time Integration and Lumped Mass Matrix **67**

4.7 Postprocessing Demonstration **68**

4.8 One-Dimensional X-FEM **68**

4.8.1 Enriched Displacement 68

4.8.2 Mass Matrix 72

4.9 Summary **72**

5. Numerical Study of Two-Dimensional Fracture Problems with X-FEM **75**

5.1 Numerical Study and Precision Analysis of X-FEM **76**

5.1.1 A Half Static Crack in a Finite Plate 76

5.1.2 A Beam with Stationary Crack under Dynamic Loading 77

5.1.3 Simulation of Complex Crack Propagation 77

5.1.4 Simulation of the Interface 79

5.1.5 Interaction Between Crack and Holes 81

5.1.6 Interfacial Crack Growth in Bimaterials 83

5.2 Two-Dimensional High-Order X-FEM **84**

5.2.1 Spectral Element-Based X-FEM 84

5.2.2 Mixed-Mode Static Crack 87

5.2.3 Kalthoff's Experiment 89

5.2.4 Mode I Moving Crack 91

5.3 Crack Branching Simulation **93**

5.3.1 Crack Branching Enrichment 94

5.3.2 Branch Criteria 95

5.3.3 Numerical Examples 97

5.4 Summary **101**

6. X-FEM on Continuum-Based Shell Elements **103**

6.1 Introduction **104**

6.2 Overview of Plate and Shell Fracture Mechanics **104**

6.2.1 Kirchhoff Plate and Shell Bending Fracture Theory 105

6.2.2 Reissner Plate and Shell Bending Fracture Theory 109

6.3 Plate and Shell Theory Applied In Finite Element Analysis **113**

6.4 Brief Introduction to General Shell Elements **115**

6.4.1 Belytschko–Lin–Tsay Shell Element 115

6.4.2 Continuum-Based Shell Element 116

6.5 X-FEM on CB Shell Elements **119**

6.5.1 Shape Function of a Crack Perpendicular to the Mid-Surface 119
6.5.2 Shape Function of a Crack Not Perpendicular to the Mid-Surface 123
6.5.3 Total Lagrangian Formulation 125
6.5.4 Time Integration Scheme and Linearization 127
6.5.5 Continuum Element Transformed to Shell 128
6.6 Crack Propagation Criterion 129
6.6.1 Stress Intensity Factor Computation 129
6.6.2 Maximum Energy Release Rate Criterion 133
6.7 Numerical Examples 136
6.7.1 Mode I Central Through-Crack in a Finite Plate 136
6.7.2 Mode III Crack Growth in a Plate 137
6.7.3 Steady Crack in a Bending Pipe 137
6.7.4 Crack Propagation Along a Given Path in a Pipe 139
6.7.5 Arbitrary Crack Growth in a Pipe 140
6.8 Summary 140

7. Subinterfacial Crack Growth in Bimaterials 143
7.1 Introduction 143
7.2 Theoretical Solutions of Subinterfacial Fracture 144
7.2.1 Complex Variable Function Solution for Sub-interfacial Cracks 144
7.2.2 Solution Considering the Crack Surface Affected Area 147
7.2.3 Analytical Solution of a Finite Dimension Structure 149
7.3 Simulation of Subinterfacial Cracks Based On X-FEM 153
7.3.1 Experiments on Subinterfacial Crack Growth 153
7.3.2 X-FEM Simulation of Subinterfacial Crack Growth 155
7.4 Equilibrium State of Subinterfacial Mode I Cracks 158
7.4.1 Effect on Fracture Mixed Level by Crack Initial Position 158
7.4.2 Effect on Material Inhomogeneity and Load Asymmetry 159
7.5 Effect on Subinterfacial Crack Growth from a Tilted Interface 163
7.6 Summary 165

8. X-FEM Modeling of Polymer Matrix Particulate/Fibrous Composites 167
8.1 Introduction 167
8.2 Level Set Method for Composite Materials 169
8.2.1 Level Set Representation 169
8.2.2 Enrichment Function 172
8.2.3 Lumped Mass Matrix 173
8.3 Microstructure Generation 175
8.4 Material Constitutive Model 176

8.5 **Numerical Examples** **177**
8.5.1 Static Analysis 177
8.5.2 Dynamic Analysis 181
8.6 **Summary** **187**

9. X-FEM Simulation of Two-Phase Flows **189**
9.1 **Governing Equations and Interfacial Conditions** **189**
9.2 **Interfacial Description of Two-Phase Flows** **192**
9.3 **X-FEM and Unknown Parameters Discretization** **194**
9.4 **Discretization of Governing Equations** **200**
9.5 **Numerical Integral Method** **205**
9.6 **Examples and Analyses** **207**
9.7 **Summary** **211**

10. Research Progress and Challenges of X-FEM **213**
10.1 **Research on Micro-Scale Crystal Plasticity** **213**
10.1.1 Discrete Dislocation Plasticity Modeling 215
10.1.2 X-FEM Simulation of Dislocations 219
10.2 **Application of Multi-Scale Simulation** **223**
10.3 **Modeling of Deformation Localization** **224**
10.4 **Summary** **228**

Appendix A: Westergaard Stress Function Method 229
Appendix B: *J* Integration 245
References 259
Index 269

Chapter 1

Overview of Extended Finite Element Method

Chapter Outline

1.1 Significance of Studying Computational Fracture Mechanics 1
1.2 Introduction to X-FEM 2
1.3 Research Status and Development of X-FEM 8
1.3.1 The Development of X-FEM Theory 8
1.3.2 Development of 3D X-FEM 10
1.4 Organization of this Book 11

1.1 SIGNIFICANCE OF STUDYING COMPUTATIONAL FRACTURE MECHANICS

Fracture is one of the most important failure modes. In various engineering fields, many catastrophic accidents have started from cracks or ends at crack propagation, such as the cracking of geologic structures and the collapse of engineering structures during earthquakes, damage of traffic vehicles during collisions, the instability crack propagation of pressure pipes, and the fracture of mechanical components. These accidents have caused great loss to people's lives and economic property. However, usually it is very difficult to quantitatively provide the causes of crack initiation. So research on fracture mechanics, which is mainly focused on studying the propagation or arrest of initiated cracks, is of great theoretical importance and has broad application potential.

Modern fracture mechanics has been booming and has been studied extensively in recent years; this is because it is already deeply rooted in the modern high-technology field and engineering applications. For example, large-scale computers facilitate the numerical simulation of complicated fracture processes, and new experimental techniques provided by modern physics, such as advanced scanning electron microscope (SEM) analysis, surface analysis, and high-speed photography, make it possible to study the fracture process from the micro-scale to the macro-scale. This understanding of the basic laws of fracture plays an important role in theoretically guiding the applications of fracture mechanics in engineering, such as the toughening of new materials, the development of biological and biomimetic materials, the seismic design of buildings and nuclear reactors, the reliability of microelectronic components, earthquake prediction in geomechanics, the exploration and storage of oil and gas, the new design of aerospace vehicles, etc. After integration with modern science and high-technology methods, fracture mechanics is taking on a new look.

Cracks in reality are usually in three dimensions, and have complicated geometries and arbitrary propagation paths. For a long time, one of the difficult challenges of mechanics has been to study crack propagation along curved or kinked paths in three-dimensional structures. In these situations, the "straight crack" assumption in conventional fracture mechanics is no longer valid, so theoretical methods are very limited for this problem. Experiments are another important way to study the propagation of curved cracks, but most results are empirical and phenomenological, and mainly focus on planar cracks. In recent decades, numerical simulations have developed rapidly along with the development of computer technology. The new progress in computational mechanics methods, such as the finite element method, boundary element method, etc., provides the possibility of solving the propagation of curved cracks. Modeling crack propagation in three-dimensional solids and curved surfaces has become one of the hottest topics in computational mechanics. Computational fracture mechanics methods roughly include the finite element method with adaptive mesh (Miehe and Gürses, 2007), nodal force release method (Zhuang and O'Donoghue, 2000a, b), element cohesive model (Xu and Needleman, 1994), and embedded discontinuity model (Belytschko et al., 1988). All of these methods have some limitations when dealing with cracks with complicated geometries, such as when the crack path needs to be predefined, the crack must propagate along the element boundary, the computational cost is high, etc. In the last decades, the extended finite element method (X-FEM) proposed in the late 1990s has become one of the most efficient tools for numerical solution of complicated fracture problems.

1.2 INTRODUCTION TO X-FEM

One of the greatest contributions the scientists made to mankind in the twentieth century was the invention of the computer, which has greatly promoted the development of related industry and scientific research. Taking computational mechanics as an example, many new methods, such as the finite element method, finite difference, and finite volume methods have rapidly developed as the invention of computer. Thanks to these methods, a lot of traditional problems in mechanics can be simulated and analyzed numerically; more importantly, a number of engineering and scientific problems can be modeled and solved. As the development of modern information technology and computational science continues, simulation-based engineering and science has become helpful to scientists in exploring the mysteries of science, and provides an effective tool for the engineer to implement engineering innovations or product development with high reliability. The finite element method (FEM) is just one of the powerful tools of simulation-based engineering and science.

Since the appearance of the first FEM paper in the mid-1950s, many papers and books on this issue have been published. Some successful experimental reports and a series of articles have made great contributions to the development of FEM. From the 1960s, with the emergence of finite element software and its rapid applications, FEM has had a huge impact on computer-aided engineering analysis. The appearance of numerous advanced software not only meets the requirement of simulation-based engineering and science, but also promotes further development of the finite element method itself. If we compare a finite element to a large tree, it is like the growth of several important branches, like hybrid elements, boundary elements, the meshless method, extended finite elements, etc., make this particular tree prosper.

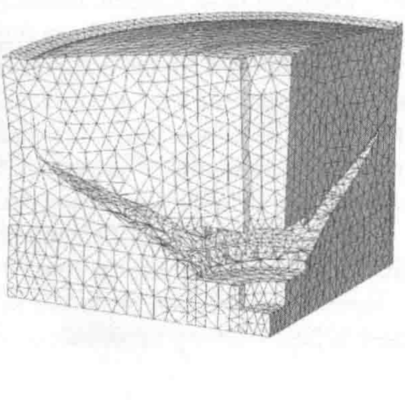
In analysis by the conventional finite element method, the physical model to be solved is divided into a series of elements connected in a certain arrangement, usually called the "mesh". However, when there are some internal defects, like interfaces, cracks, voids, inclusions, etc. in the domain, it will create some difficulties in the meshing process. On one hand, the element boundary must coincide with the geometric edge of the defects, which will induce some distortion in the element; on the other hand, the mesh size will be dependent on the geometric size of the small defects, leading to a nonuniform mesh distribution in which the meshes around the defects are dense, while those far from defects are sparse. As we know, the smallest mesh size decides the critical stable time increment in explicit analysis. So the small elements around the defects will heavily increase the computational cost. Also, defects, like cracks, can only propagate along the element edge, and not flow along a natural arbitrary path. Aiming at solving these shortcomings by using the conventional FEM to solve crack or other defects with discontinuous interfaces, Belytschko and Moës proposed a new computational method called the "extended finite element method (X-FEM)" (Belytschko and Black, 1999; Moës et al., 1999), and made an important improvement to the foundation of conventional FEM. In the last 10 years, X-FEM has been constantly improved and developed, and has already become a powerful and promising method for dealing with complicated mechanics problems, like discontinuous field, localized deformation, fracture, and so on. It has been widely used in civil engineering, aviation and space, material science, etc.

The core idea of X-FEM is to use a discontinuous function as the basis of a shape function to capture the jump of field variables (e.g., displacement) in the computational domain. So in the calculations, the description for the discontinuous field is totally mesh-independent. It is this advantage that makes it very suitable for dealing with fracture problems. Figure 1.1 is an example of a three-dimensional fracture simulated by X-FEM (Areias and Belytschko, 2005b), in which we find that the crack surface and front are independent of the mesh. Figure 1.2 demonstrates the process of transition crack growth under impact loading on the left lower side of a plane plate, in which we can investigate impact wave propagation in the plate and the stress singularity field at the crack tip location. In addition, it is very convenient to model the crack with complex geometries using X-FEM; one example of crack branching simulation is given in Figure 1.3 (Xu et al., 2013).

X-FEM is not only used to simulate cracks, but also to simulate heterogeneous materials with voids and inclusions (Belytschko et al., 2003b; Sukumar et al., 2001). The main difference is that: for cracks, the discontinuous field at the crack surface is the displacement; for inclusions, the derivative of displacement with respect to a spatial coordinate — the strain — is discontinuous. These two situations are defined as strong discontinuity (jump of displacement field) and weak discontinuity (jump of derivative of displacement with respect to spatial coordinate) respectively. Two different enrichment shape functions will be used to capture the two different discontinuities. Figure 1.4 shows an example of studying the effective modulus of carbon nanotube composites by X-FEM modeling. In the simulations, the mesh boundary does not have to coincide with the material interfaces, so the representative volume element (RVE) can be meshed by brick elements, which will greatly increase the efficiency of modeling.

The other advantage of X-FEM is that it can make use of known analytical solutions to construct the shape function basis, so accurate results can be obtained even using a relatively coarse mesh. When applying conventional FEM to model the singular field, like the stress field near a crack tip or dislocation core, a very dense mesh has to be used. However, in X-FEM, by introducing the known displacement solution of cracks or dislocations into the enrichment shape function, a satisfied solution can be obtained under a relatively coarse mesh. Figure 1.5 shows a plate with an initial crack at the left

FIGURE 1.1 X-FEM modeling of 3D fracture: displacement is magnified 200 times (Areias and Belytschko, 2005b).



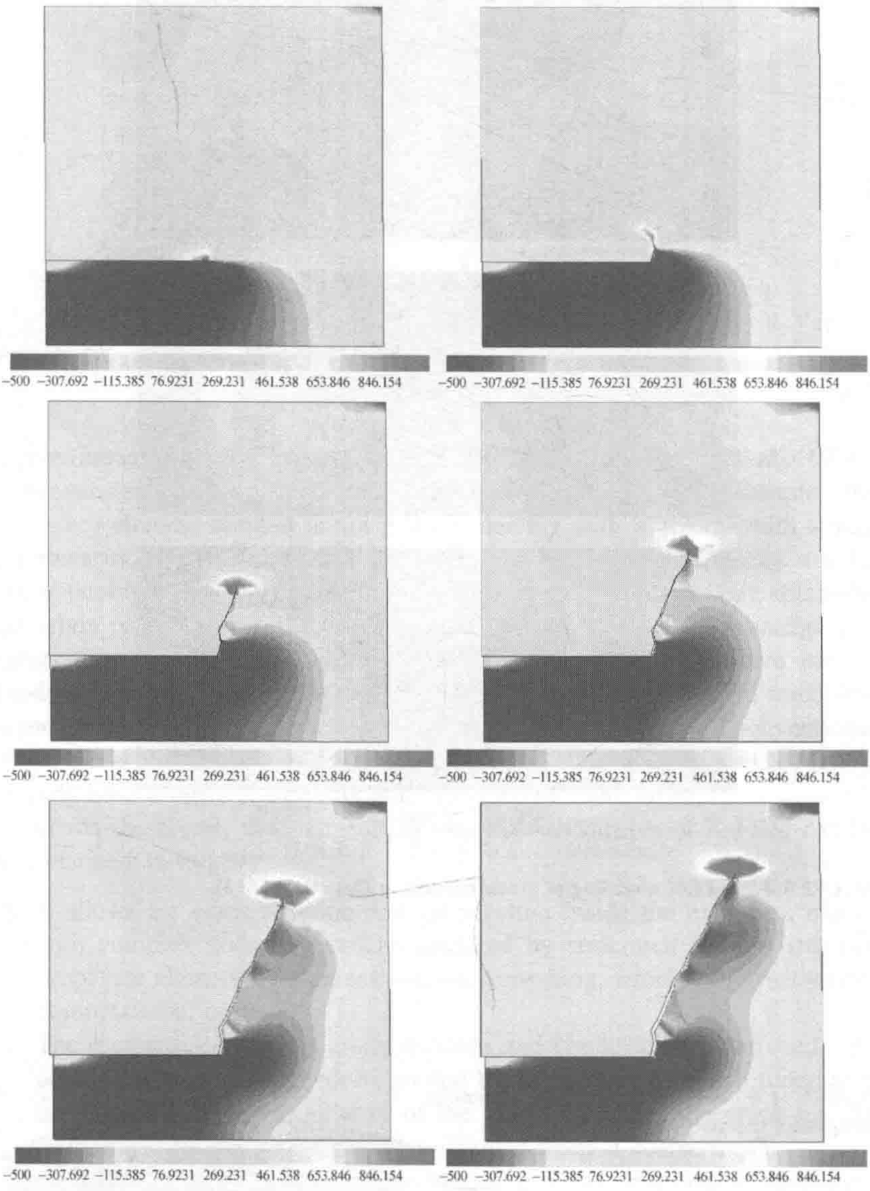


FIGURE 1.2 **Process of transition crack growth under impact on the left lower side.**

edge; the stress intensity factor can be calculated as a function of crack length. In the X-FEM simulation, without using the fine mesh near the crack tip, the calculated stress intensity factor (SIF) for 41 by 41 uniform elements can compare well with analytic solution.

It is worth pointing out that, other methods, like the boundary element method and meshless method, also have important applications on solving

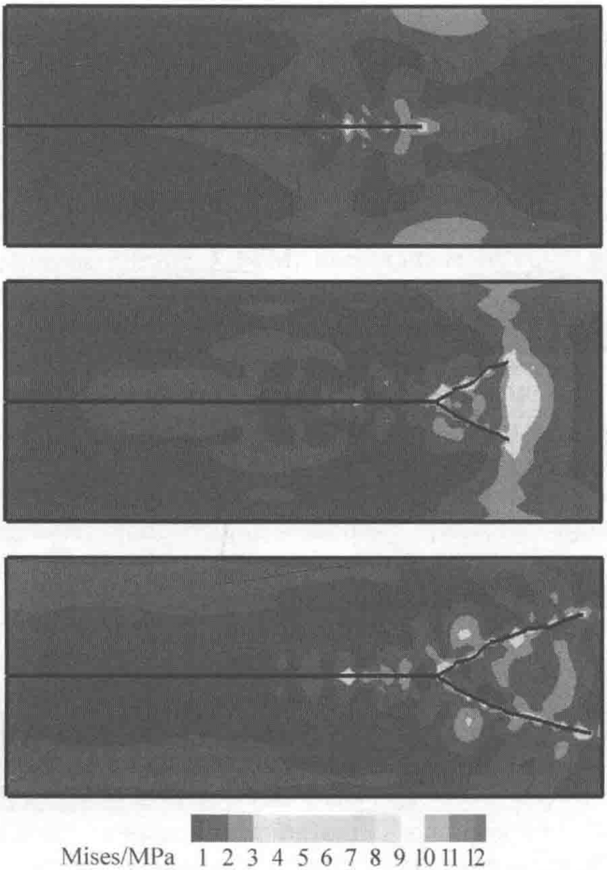
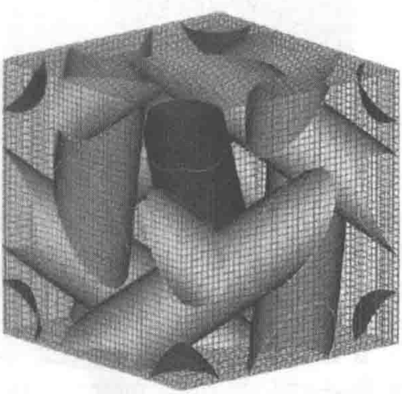


FIGURE 1.3 X-FEM modeling of crack branching (Xu et al., 2013).

FIGURE 1.4 X-FEM model of nanotube composites (Belytschko et al., 2003b).



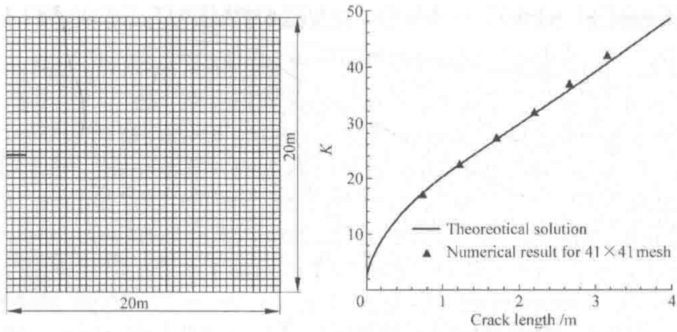


FIGURE 1.5 Stress intensity factor of a static crack in a finite plate.

discontinuous problems (Blandford et al., 1981; Belytschko et al., 1994). However, some inherent flaws limit their promotion: for example, the boundary element method is not good at dealing with problems with strong nonlinearity, heterogeneity, and so on; the meshless method lacks a solid theoretical foundation and rigorous mathematical proof, so there are still some uncertain parameters like the radius of the interpolation domain, background integration domain, etc.; commercial software still does not have mature modules for these two methods. In contrast, X-FEM is developed under the standard framework of FEM, and retains all the advantages of the conventional FEM method. Some commercial software, like ABAQUS and LS-DYNA, already have a basic X-FEM module for fracture simulations.

Given the above, the characterizations and advantages of X-FEM can be summarized as follows:

1. It allows for crack location and propagation inside the elements; cracks with complex geometry can be modeled by structured meshes and can propagate element by element without remeshing, which will greatly save computational cost.
2. The elements containing crack surfaces and crack tips are enriched with additional degrees of freedom, so that the discontinuous shape function is used to capture the singularity of the stress field near the crack tip. An accurate solution can therefore be obtained using a coarse mesh.
3. Compared with the remeshing technique in FEM, mapping of field variables after crack propagation is not necessary in X-FEM.
4. Compared with the boundary element method, X-FEM is applicable for multi-material or multi-phase problems, especially problems with geometric and contact nonlinearities.
5. It is convenient to implement in commercial software and with parallel computing.

All the features above illustrate why X-FEM has many successful applications.