YANJIUSHENG RUXUE KAOSHI SHUXUE FENXI ZHENTI JIJIE(ZHONG CE)

研究生入学考试数学分析真题集解（中册）

梁志清 黄军华 钟镇权 ◎ 编著

SWJTUP 西南交通大学出版社

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目　　录

4 一元函数积分学

4.1 不定积分计算

1. 计算下列积分.

(1) $\int \sqrt{1+\cos x}\,dx$.(武汉大学 2011)

(2) $\int \frac{\sec^2 x}{4+\tan^2 x}dx$.(太原科技 2008)

(3) $\int \frac{dx}{\cos^4 x \sin^4 x}$.(上海师大 2000)

(4) $\int \frac{dx}{1+\tan x}$.(武汉大学 2010,复旦大学 1997)

解题过程:

(1) $\int \sqrt{1+\cos x}\,dx = \int \sqrt{2\cos^2 \frac{x}{2}}dx = \sqrt{2}\int \left|\cos\frac{x}{2}\right| dx = \pm 2\sqrt{2}\sin\frac{x}{2}+C$.

(2) $\int \frac{\sec^2 x}{4+\tan^2 x}dx = \int \frac{1}{4+\tan^2 x}d\tan x = \frac{1}{2}\arctan\left(\frac{\tan x}{2}\right)+C$.

(3) $\int \frac{dx}{\cos^4 x\sin^4 x} = 16\int \frac{dx}{\sin^4 2x} = -8\int (1+\cot^2 2x)d\cot 2x = -8\cot 2x - \frac{8}{3}\cot^3 2x + C$.

(4) 方法 1: $\int \frac{dx}{1+\tan x} = \int \frac{\cos x}{\cos x+\sin x}dx = \frac{1}{2}\int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{\cos x+\sin x}dx$

$$= \frac{1}{2}x + \frac{1}{2}\ln|\sin x+\cos x|+C .$$

方法 2:记 $I = \int \frac{\cos x}{\cos x+\sin x}dx$, $J = \int \frac{\sin x}{\cos x+\sin x}dx$,则

$$I-J = \int \frac{\cos x-\sin x}{\cos x+\sin x}dx = \int \frac{d(\cos x+\sin x)}{\cos x+\sin x} = \ln|\cos x+\sin x|+C ,\ I+J = \int dx = x+C .$$

所以 $$I = \int \frac{\cos x}{\cos x+\sin x}dx = \frac{x+\ln|\cos x+\sin x|}{2}+C ,$$

$$J = \int \frac{\sin x}{\cos x+\sin x}dx = \frac{x-\ln|\cos x+\sin x|}{2}+C .$$

2. 计算下列积分.

(1) $\int \frac{\cos x\sin^3 x}{1+\cos^2 x}dx$.(浙江师大 2011,青岛大学 2011,华东师大 2000,天津大学 2009)

(2) $\int \frac{\sin x\cos^3 x}{1+\sin^2 x}dx$.(湖南农大 2011,西北师大 2005,广西民大 2011/2014)

(3) $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}(ab\neq 0)$.(太原理工 2006,扬州大学 2004 ($a=1,b=\sqrt{2}$))

(4) $\int \frac{dx}{\tan x+\sin x}$.(新疆大学 2009)

(5) $\int \frac{dx}{\cos 2x\sin x}$.(福建师大 2005)

解题过程:

(1) 令 $t=\cos x$,则

$$\int\frac{\cos x\sin^3 x}{1+\cos^2 x}\mathrm{d}x=-\int\frac{\cos x(1-\cos^2 x)}{1+\cos^2 x}\mathrm{d}(\cos x)=\int\frac{t(t^2-1)}{1+t^2}\mathrm{d}t=\int\frac{t(1+t^2)-2t}{1+t^2}\mathrm{d}t$$

$$=\int\left(t-\frac{2t}{1+t^2}\right)\mathrm{d}t=\frac{t^2}{2}-\ln(1+t^2)+C=\frac{1}{2}\cos^2 x-\ln(1+\cos^2 x)+C.$$

(2) $$\int\frac{\sin x\cos^3 x}{1+\sin^2 x}\mathrm{d}x=\int\frac{\sin x\cos^2 x}{1+\sin^2 x}\mathrm{d}(\sin x)=\int\frac{\sin x(1-\sin^2 x)}{1+\sin^2 x}\mathrm{d}(\sin x)=\frac{1}{2}\int\frac{1-\sin^2 x}{1+\sin^2 x}\mathrm{d}(\sin^2 x)$$

$$=\frac{1}{2}\int\frac{2-(1+\sin^2 x)}{1+\sin^2 x}\mathrm{d}(\sin^2 x)=\frac{1}{2}\int\left(\frac{2}{1+\sin^2 x}-1\right)\mathrm{d}(\sin^2 x)$$

$$=\int\frac{1}{1+\sin^2 x}\mathrm{d}(1+\sin^2 x)-\frac{1}{2}\int\mathrm{d}(\sin^2 x)=\ln(1+\sin^2 x)-\frac{1}{2}\sin^2 x+C.$$

(3) 令 $t=\tan x$,则

$$\int\frac{\mathrm{d}x}{a^2\sin^2 x+b^2\cos^2 x}=\int\frac{\sec^2 x\mathrm{d}x}{a^2\tan^2 x+b^2}=\int\frac{\mathrm{d}(\tan x)}{a^2\tan^2 x+b^2}=\int\frac{\mathrm{d}t}{a^2t^2+b^2}=\frac{1}{a}\int\frac{\mathrm{d}(at)}{(at)^2+b^2}$$

$$=\frac{1}{ab}\arctan\frac{at}{b}+C=\frac{1}{ab}\arctan\left(\frac{a\tan x}{b}\right)+C.$$

(4) $$\int\frac{\mathrm{d}x}{\tan x+\sin x}=\int\frac{\cos x\mathrm{d}x}{\sin x(1+\cos x)}=-\int\frac{\cos x\mathrm{d}\cos x}{(1-\cos^2 x)(1+\cos x)}$$

$$=-\frac{1}{2}\int\frac{(1+\cos x)-(1-\cos x)}{(1-\cos^2 x)(1+\cos x)}\mathrm{d}\cos x=-\frac{1}{2}\int\frac{\mathrm{d}\cos x}{1-\cos^2 x}+\frac{1}{2}\int\frac{\mathrm{d}\cos x}{(1+\cos x)^2}$$

$$=\frac{1}{4}\ln\left|\frac{1-\cos x}{1+\cos x}\right|-\frac{1}{2(1+\cos x)}+C=\frac{1}{2}\ln\left|\tan\frac{x}{2}\right|-\frac{1}{4}\tan^2\frac{x}{2}+C.$$

(5) 令 $t=\cos x$,则

$$\int\frac{\mathrm{d}x}{\cos 2x\sin x}=\int\frac{\sin x}{(2\cos^2 x-1)\sin^2 x}\mathrm{d}x=-\int\frac{1}{(2t^2-1)(1-t^2)}\mathrm{d}t$$

$$=\int\frac{1}{t^2-1}\mathrm{d}t-\sqrt{2}\int\frac{1}{(\sqrt{2}t)^2-1}\mathrm{d}(\sqrt{2}t)=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|-\sqrt{2}\ln\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right|+C$$

$$=\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|-\sqrt{2}\ln\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right|+C.$$

3. 求下列积分.

(1) $\int\frac{\mathrm{d}x}{1+\sin x}$.(西南大学 2003,温州大学 2012)　　(2) $\int\frac{\mathrm{d}x}{1+4\cos x}$.(地质大学 2002)

(3) $\int\frac{\mathrm{d}x}{\cos x\sin^3 x}$.(河南师大 2011,郑州大学 2001)

(4) $\int\frac{\mathrm{d}x}{2+\cos x+\sin x}$.(杭州师大 2014)

(5) $\int \frac{5\sin x + 2\cos x}{\sin x + 3\cos x} dx$.(北京科技 2011/2004)

(6) $\int \frac{\ln(2+\sin^2 x)}{(1+\sin^2 x)^2} \sin 2x dx$.(西北师大 2002)

(7) $\int \frac{\sqrt{1+\sin x}}{\cos x} dx$.(南航 2014)

解题过程:

(1) 方法 1:由于 $\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2 \frac{x}{2}} = \int \frac{1}{\cos^2 \frac{x}{2}} d\left(\frac{x}{2}\right) = \tan\frac{x}{2} + C$,所以

$$\int \frac{dx}{1+\sin x} = -\int \frac{1}{1+\cos\left(\frac{\pi}{2}-x\right)} d\left(\frac{\pi}{2}-x\right) = -\tan\left[\frac{1}{2}\left(\frac{\pi}{2}-x\right)\right] + C = -\tan\left(\frac{\pi}{4}-\frac{x}{2}\right) + C .$$

方法 2: $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)dx}{1-\sin^2 x} = \int \frac{dx}{\cos^2 x} + \int \frac{d\cos x}{\cos^2 x} = \tan x - \frac{1}{\cos x} + C$.

方法 3: $\int \frac{dx}{1+\sin x} = \int \frac{dx}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2} = \int \frac{dx}{\cos^2\frac{x}{2}\cdot\left(\tan\frac{x}{2}+1\right)^2} = 2\int \frac{d\tan\frac{x}{2}}{\left(\tan\frac{x}{2}+1\right)^2} = \frac{-2}{\tan\frac{x}{2}+1} + C$.

方法 4:令 $t = \tan\frac{x}{2}$,则 $\int \frac{dx}{1+\sin x} = \int \frac{1}{1+\frac{2t}{1+t^2}} \frac{2}{1+t^2} dt = 2\int \frac{1}{(t+1)^2} dt = \frac{-2}{t+1} + C = \frac{-2}{\tan\frac{x}{2}+1} + C$.

(2) 令 $t = \tan\frac{x}{2}$,则

$$\int \frac{dx}{1+4\cos x} = \int \frac{1}{1+4\frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = 2\int \frac{1}{5-3t^2} dt = -\frac{1}{\sqrt{5}} \int \left(\frac{1}{\sqrt{3}t-\sqrt{5}} - \frac{1}{\sqrt{3}t+\sqrt{5}}\right) dt$$

$$= \frac{1}{\sqrt{15}} \ln\frac{\sqrt{3}t+\sqrt{5}}{\sqrt{3}t-\sqrt{5}} + C = \frac{1}{\sqrt{15}} \ln\frac{\sqrt{3}\tan\frac{x}{2}+\sqrt{5}}{\sqrt{3}\tan\frac{x}{2}-\sqrt{5}} + C.$$

(3) $\int \frac{dx}{\cos x\sin^3 x} = \int \frac{\cos^2 x+\sin^2 x}{\cos x\sin^3 x} dx = \int \frac{\cos x}{\sin^3 x} dx + \int \frac{1}{\cos x\sin x} dx = \int \frac{d\sin x}{\sin^3 x} + \int \csc 2x d(2x)$

$$= -\frac{1}{2}\sin^2 x + \ln|\csc 2x - \cot 2x| + C .$$

(4) 令 $t = \tan\frac{x}{2}$,则

$$\int \frac{dx}{2+\cos x+\sin x} = \int \frac{1}{2+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \frac{2}{1+t^2} dt = 2\int \frac{1}{(t+1)^2+2} dt = \sqrt{2}\arctan\frac{t+1}{\sqrt{2}} + C$$

$$=\sqrt{2}\arctan\left(\frac{1}{\sqrt{2}}\left(\tan\frac{x}{2}+1\right)\right)+C.$$

(5) 设

$$\int\frac{5\sin x+2\cos x}{\sin x+3\cos x}\mathrm{d}x=\int\frac{A(\sin x+3\cos x)+B(\sin x+3\cos x)'}{\sin x+3\cos x}\mathrm{d}x=Ax+B\ln|\sin x+3\cos x|+C,$$

其中 A,B 待定.

由 $5\sin x+2\cos x=A(\sin x+3\cos x)+B(\cos x-3\sin x)$ 得 $A=\frac{11}{10},B=-\frac{13}{10}$.于是

$$\int\frac{5\sin x+2\cos x}{\sin x+3\cos x}\mathrm{d}x=\int\frac{A(\sin x+3\cos x)+B(\sin x+3\cos x)'}{\sin x+3\cos x}\mathrm{d}x=\frac{11}{10}x-\frac{13}{10}\ln|\sin x+3\cos x|+C.$$

(6) 令 $t=\sin^2 x$,则

$$\int\frac{\ln(2+\sin^2 x)}{(1+\sin^2 x)^2}\sin 2x\mathrm{d}x=\int\frac{\ln(2+t)}{(1+t)^2}\mathrm{d}t=-\int\ln(2+t)\mathrm{d}\left(\frac{1}{1+t}\right)=-\frac{\ln(2+t)}{1+t}+\int\frac{1}{(1+t)(2+t)}\mathrm{d}t$$

$$=-\frac{\ln(2+t)}{1+t}+\ln\frac{1+t}{2+t}+C=-\frac{\ln(2+\sin^2 x)}{1+\sin^2 x}+\ln\frac{1+\sin^2 x}{2+\sin^2 x}+C.$$

(7) 令 $t=\sqrt{1+\sin x}$,则

$$\int\frac{\sqrt{1+\sin x}}{\cos x}\mathrm{d}x=\int\frac{2t^2}{2t^2-t^4}\mathrm{d}t=2\int\frac{1}{2-t^2}\mathrm{d}t=\frac{1}{\sqrt{2}}\ln\frac{|t+\sqrt{2}|}{|t-\sqrt{2}|}+C=\frac{1}{\sqrt{2}}\ln\frac{|\sqrt{1+\sin x}+\sqrt{2}|}{|\sqrt{1+\sin x}-\sqrt{2}|}+C.$$

4. 求下列积分.

(1) $\int\frac{\arctan x}{x^2(1+x^2)}\mathrm{d}x$.(地质大学 2006,华东水电 2009) (2) $\int\frac{\arctan\sqrt{x}}{\sqrt{x}+\sqrt{x^3}}\mathrm{d}x$.(河南师大 2009)

(3) $\int\frac{\arctan x}{x^2}\mathrm{d}x$.(人民大学 2006) (4) $\int\frac{\arctan \mathrm{e}^x}{\mathrm{e}^x}\mathrm{d}x$.(湖南农大 2010,广西师大 2009)

解题过程:

(1) $$\int\frac{\arctan x}{x^2(1+x^2)}\mathrm{d}x=\int\frac{\arctan x}{x^2}\mathrm{d}x-\int\frac{\arctan x}{1+x^2}\mathrm{d}x=-\frac{1}{2}\arctan^2 x-\int\arctan x\mathrm{d}\left(\frac{1}{x}\right)$$

$$=-\frac{1}{2}\arctan^2 x-\frac{1}{x}\arctan x+\int\frac{1}{x}\cdot\frac{1}{1+x^2}\mathrm{d}x$$

$$=-\frac{1}{2}\arctan^2 x-\frac{1}{x}\arctan x+\frac{1}{2}\int\frac{1}{x^2}\cdot\frac{1}{1+x^2}\mathrm{d}(x^2)$$

$$=-\frac{\arctan x}{x}-\frac{1}{2}\arctan^2 x+\frac{1}{2}\ln\frac{x^2}{1+x^2}+C.$$

(2) $$\int\frac{\arctan\sqrt{x}}{\sqrt{x}+\sqrt{x^3}}\mathrm{d}x=\int\frac{\arctan\sqrt{x}}{\sqrt{x}(1+x)}\mathrm{d}x=2\int\frac{\arctan\sqrt{x}}{1+x}\mathrm{d}(\sqrt{x})$$

$$=2\int\arctan\sqrt{x}\mathrm{d}(\arctan\sqrt{x})=\arctan^2\sqrt{x}+C.$$

(3) $$\int\frac{\arctan x}{x^2}\mathrm{d}x=-\int\arctan x\,\mathrm{d}\left(\frac{1}{x}\right)=-\frac{\arctan x}{x}+\int\frac{1}{x(1+x^2)}\mathrm{d}x$$

$$= -\frac{\arctan x}{x} + \int\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx = -\frac{\arctan x}{x} + \ln|x| - \frac{1}{2}\ln(1+x^2) + C.$$

(4) 方法 1:令 $u = e^x$,则

$$\int \frac{\arctan e^x}{e^x}dx = \int \frac{\arctan e^x}{e^{2x}}d(e^x) = \int \frac{\arctan u}{u^2}du = -\frac{\arctan u}{u} + \ln|u| - \frac{1}{2}\ln(1+u^2) + C$$

$$= -e^{-x}\arctan e^x + x - \frac{1}{2}\ln(1+e^{2x}) + C.$$

方法 2: $\int \frac{\arctan e^x}{e^x}dx = -e^{-x}\arctan e^x + \int e^{-x}\cdot\frac{e^x}{1+e^{2x}}dx = -e^{-x}\arctan e^x + \int \frac{1}{1+e^{2x}}dx$.

又

$$\int \frac{1}{1+e^{2x}}dx \xlongequal{e^x=t} \int \frac{1}{1+t^2}\cdot\frac{1}{t}dt = \int \frac{1}{1+t^2}\cdot\frac{1}{t^2}\cdot t dt = \frac{1}{2}\int\left(\frac{1}{t^2} - \frac{1}{1+t^2}\right)dt^2$$

$$= \frac{1}{2}\ln\frac{t^2}{1+t^2} + C = x - \frac{1}{2}\ln(1+e^{2x}) + C,$$

所以

$$\int \frac{\arctan e^x}{e^x}dx = -e^{-x}\arctan e^x + x - \frac{1}{2}\ln(1+e^{2x}) + C.$$

5. 求下列积分.

(1) $\int \arctan 2x dx$.(湖南师大 2009)

(2) $\int x\arctan x dx$.(广西民大 2009,河海大学 2002,湖南师大 1998)

(3) $\int \arctan\sqrt{x} dx$.(华侨大学 2012,兰州大学 2009,山西师大 2010)

解题过程:

(1) $\int \arctan 2x dx = \frac{1}{2}\int \arctan 2x d(2x) = \frac{1}{2}\left[2x\arctan 2x - 2\int \frac{2x}{1+(2x)^2}dx\right]$

$$= x\arctan 2x - \ln(1+(2x)^2) + C = x\arctan 2x - \ln(1+4x^2) + C.$$

(2) $\int x\arctan x dx = \frac{1}{2}\int \arctan x d(x^2) = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}(x - \arctan x) + C = \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C.$$

(3) 令 $t = \sqrt{x}$,则

$$\int \arctan\sqrt{x}dx = \int \arctan t d(t^2+1) = (t^2+1)\arctan t - \int dt$$

$$= (t^2+1)\arctan t - t + C = (x+1)\arctan\sqrt{x} - \sqrt{x} + C.$$

6. 求下列积分.

(1) $\int \sqrt{x}\sin\sqrt{x}dx$.(中山大学 2008)　　(2) $\int \cos^2\sqrt{x}dx$.(新疆大学 2005/2006)

(3) $\int x\sin ax\cos bx dx(a \neq 0, b \neq 0, a^2 \neq b^2)$.(北京交大 2004)

解题过程:

(1) 令 $t = \sqrt{x}$,则

$$\int\sqrt{x}\sin\sqrt{x}\mathrm{d}x=2\int t^2\sin t\mathrm{d}t=-2\int t^2\mathrm{d}(\cos t)=-2t^2\cos t+4\int t\cos t\mathrm{d}t=-2t^2\cos t+4\int t\mathrm{d}\sin t$$
$$=-2t^2\cos t+4t\sin t+4\cos t=(4-2x)\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+C.$$

(2) 令 $t=\sqrt{x}$,则

$$\int\cos^2\sqrt{x}\mathrm{d}x=\int 2t\cos^2 t\mathrm{d}t=\int t(1+\cos 2t)\mathrm{d}t=\frac{t^2}{2}+\frac{1}{2}\int t\mathrm{d}\sin 2t$$
$$=\frac{t^2}{2}+\frac{1}{2}\left(t\sin 2t-\int\sin 2t\mathrm{d}t\right)=\frac{t^2}{2}+\frac{1}{2}\left(t\sin 2t+\frac{1}{2}\cos 2t\right)+C$$
$$=\frac{1}{4}(2x+2\sqrt{x}\sin 2\sqrt{x}+\cos 2\sqrt{x})+C.$$

(3) $$\int x\sin ax\cos bx\mathrm{d}x=\frac{1}{2}\int x[\sin(a-b)x+\sin(a+b)x]\mathrm{d}x=\frac{1}{2}\int x\mathrm{d}\left(-\frac{\cos(a-b)x}{a-b}-\frac{\cos(a+b)x}{a+b}\right)$$
$$=\frac{1}{2}x\left(-\frac{\cos(a-b)x}{a-b}-\frac{\cos(a+b)x}{a+b}\right)-\frac{1}{2}\int\left(-\frac{\cos(a-b)x}{a-b}-\frac{\cos(a+b)x}{a+b}\right)\mathrm{d}x$$
$$=\frac{1}{2}x\left(-\frac{\cos(a-b)x}{a-b}-\frac{\cos(a+b)x}{a+b}\right)+\frac{1}{2}\left(\frac{\sin(a-b)x}{(a-b)^2}+\frac{\sin(a+b)x}{(a+b)^2}\right)+C.$$

7. 求下列积分.

(1) $\int x\arcsin x\mathrm{d}x$.(浙江师大 2013) (2) $\int\frac{\arcsin\sqrt{x}}{\sqrt{x}}\mathrm{d}x$.(杭州师大 2009,西南大学 2005)

(3) $\int\frac{\arcsin\mathrm{e}^x}{\mathrm{e}^x}\mathrm{d}x$.(中山大学 2007,湖南农大 2010,河北大学 2008)

(4) $\int\frac{\arccos x}{\sqrt{(1-x^2)^3}}\mathrm{d}x$.(杭州师大 2010)

(5) $\int\frac{x\arccos x}{\sqrt{1-x^2}}\mathrm{d}x$.(华东理工 2001,湖南师大 2000)

解题过程:

(1) $$\int x\arcsin x\mathrm{d}x=\frac{1}{2}\int\arcsin x\mathrm{d}(x^2)=\frac{1}{2}\left(x^2\arcsin x-\int x^2\frac{1}{\sqrt{1-x^2}}\mathrm{d}x\right).$$

$$\int\frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x\xlongequal{x=\sin t}\int\frac{\sin^2 t}{\cos t}\cos t\mathrm{d}t=\int\frac{1-\cos 2t}{2}\mathrm{d}t=\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)=\frac{1}{2}(\arcsin x-x\sqrt{1-x^2}).$$

所以 $$\int x\arcsin x\mathrm{d}x=\frac{1}{2}\left(x^2\arcsin x-\int x^2\frac{1}{\sqrt{1-x^2}}\mathrm{d}x\right)=\frac{1}{2}\left[x^2\arcsin x-\frac{1}{2}(\arcsin x-x\sqrt{1-x^2})\right]+C$$
$$=\frac{1}{2}x^2\arcsin x-\frac{1}{4}\arcsin x+\frac{1}{4}x\sqrt{1-x^2}+C.$$

(2) 令 $t=\sqrt{x}$,则

$$\int\frac{\arcsin\sqrt{x}}{\sqrt{x}}\mathrm{d}x=2\int\frac{\arcsin t}{t}t\mathrm{d}t=2\int\arcsin t\mathrm{d}t=2\left(t\arcsin t-\int\frac{t}{\sqrt{1-t^2}}\mathrm{d}t\right)$$

$$=2\left[t\arcsin t+\frac{1}{2}\int(1-t^2)^{-\frac{1}{2}}\mathrm{d}(1-t^2)\right]=2(t\arcsin t+\sqrt{1-t^2})+C$$

$$=2\sqrt{x}\arcsin\sqrt{x}+2\sqrt{1-x}+C.$$

(3) 令 $u=\mathrm{e}^x$,则

$$\int\frac{\arcsin\mathrm{e}^x}{\mathrm{e}^x}\mathrm{d}x=\int\frac{\arcsin\mathrm{e}^x}{\mathrm{e}^{2x}}\mathrm{d}\mathrm{e}^x=\int\frac{\arcsin u}{u^2}\mathrm{d}u=-\int\arcsin u\,\mathrm{d}u^{-1}=-\left(\frac{\arcsin u}{u}-\int\frac{1}{u\sqrt{1-u^2}}\mathrm{d}u\right)$$

$$=-\left(\frac{\arcsin u}{u}+\int\frac{1}{\sqrt{u^{-2}-1}}\mathrm{d}(u^{-1})\right)=-\left(\frac{\arcsin u}{u}+\ln\left|u^{-1}+\sqrt{u^{-2}-1}\right|\right)+C$$

$$=-\left(\frac{\arcsin\mathrm{e}^x}{\mathrm{e}^x}+\ln\left|\frac{1+\sqrt{1-\mathrm{e}^{2x}}}{\mathrm{e}^x}\right|\right)+C.$$

(4) 令 $t=\arccos x$,则 $x=\cos t$, $\mathrm{d}x=-\sin t\mathrm{d}t$,于是

$$\int\frac{\arccos x}{\sqrt{(1-x^2)^3}}\mathrm{d}x=-\int\frac{t}{\sin^3 t}\cdot\sin t\mathrm{d}t=-\int t\csc^2 t\mathrm{d}t=\int t\mathrm{d}(\cot t)=t\cot t-\int\cot t\mathrm{d}t=t\cot t-\int\frac{\mathrm{d}(\sin t)}{\sin t}$$

$$=t\cot t-\ln|\sin t|+C=\arccos x\cot(\arccos x)-\ln|\sin(\arccos x)|+C.$$

(5) $$\int\frac{x\arccos x}{\sqrt{1-x^2}}\mathrm{d}x=-2\int\arccos x\mathrm{d}\sqrt{1-x^2}=-2\left(\sqrt{1-x^2}\arccos x+\int\frac{1}{1-x^2}\mathrm{d}x\right)$$

$$=-2\left(\sqrt{1-x^2}\arccos x+\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)+C.$$

8. 设 n 为自然数,求下列不定积分的递推公式.

(1) $I_n=\int\tan^n x\mathrm{d}x$,并计算 $\int\tan^4\mathrm{d}x$.(上海理工 2005,矿业大学 2005,山东大学)

(2) $I_n=\int\sec^n x\mathrm{d}x$,并计算 $\int\sec^3 x\mathrm{d}x$.(地质大学 2004,广西师大 2012,河南大学 2001,暨南大学 2007)

(3) $I_n=\int x^n\cos x\mathrm{d}x$,并计算 $\int x^3\cos x\mathrm{d}x$.(华南师大 2008)

(4) $I_n=\int(\ln x)^n\mathrm{d}x$,并计算 $\int\ln^2 x\mathrm{d}x$.(湖北大学 2001,华东师大 2003)

解题过程:

(1) $$I_n=\int\tan^{n-2}x(\sec^2 x-1)\mathrm{d}x=\int\tan^{n-2}x\sec^2 x\,\mathrm{d}x-\int\tan^{n-2}x\,\mathrm{d}x$$

$$=\int\tan^{n-2}x\,\mathrm{d}\tan x-I_{n-2}=\frac{1}{n-1}\tan^{n-1}x-I_{n-2},$$

即 $$I_n=\frac{1}{n-1}\tan^{n-1}x-I_{n-2},\ n=2,3,4,\cdots.$$

$$\int\tan^4\mathrm{d}x=I_4=\frac{1}{3}\tan^3 x-\int\tan^2 x\mathrm{d}x=\frac{1}{3}\tan^3 x-\tan x+x+C.$$

(2) $$I_n=\int\frac{\mathrm{d}x}{\cos^n x}=\int\frac{\mathrm{d}\tan x}{\cos^{n-2}x}=\frac{\tan x}{\cos^{n-2}x}-(n-2)\int\frac{\tan x}{\cos^{n-1}x}\sin x\mathrm{d}x$$

$$=\frac{\tan x}{\cos^{n-2}x}-(n-2)\int\frac{1-\cos^2 x}{\cos^n x}\mathrm{d}x=\frac{\tan x}{\cos^{n-2}x}-(n-2)(I_n-I_{n-2}).$$

于是 $I_n=\frac{1}{n-1}\frac{\sin x}{\cos^{n-1}x}+\frac{n-2}{n-1}I_{n-2},(n=2,3,4,\cdots)$,其中 $I_0=x+C,I_1=\ln|\sec x+\tan x|+C$.

$$\int\sec^3 x\mathrm{d}x=I_3=\frac{1}{2}\frac{\sin x}{\cos^2 x}+\frac{1}{2}I_1=\frac{1}{2}(\tan x\sec x+\ln|\tan x+\sec x|)+C.$$

(3) $I_n=\int x^n\cos x\mathrm{d}x=\int x^n\mathrm{d}\sin x=x^n\sin x-n\int x^{n-1}\sin x\mathrm{d}x$

$$=x^n\sin x+nx^{n-1}\cos x-n(n-1)\int x^{n-2}\cos x\mathrm{d}x.$$

于是 $I_n=x^n\sin x+nx^{n-1}\cos x-n(n-1)I_{n-2},(n=2,3,4,\cdots)$,其中 $I_0=\sin x+C,I_1=x\sin x-\cos x+C$.

$$\int x^3\cos x\mathrm{d}x=I_3=x^3\sin x+3x^2\cos x-6I_1=x^3\sin x+3x^2\cos x-6(x\sin x-\cos x)+C.$$

(4) $I_n=\int(\ln x)^n\mathrm{d}x=x(\ln x)^n-n\int(\ln x)^{n-1}\mathrm{d}x=x(\ln x)^n-nI_{n-1}$.

$$\int(\ln x)^2\mathrm{d}x=I_2=x\ln^2 x-2I_1=x\ln^2 x-2(x\ln x-I_0)=x\ln^2 x-2(x\ln x-x)+C.$$

9. 求下列积分.

(1) $\int\mathrm{e}^{ax}\sin bx\mathrm{d}x$.(湖南师大 2010($b$=1),深圳大学 2006($a$=1,$b$=1),湘潭大学 2008($a$=1,$b$=2))

(2) $\int\cos(\ln x)\mathrm{d}x$, $\int\sin(\ln x)\mathrm{d}x$.(南京航空 2007,四川大学 2011,新疆大学 2004,南京农大 2009,复旦大学 2002)

(3) $\int\mathrm{e}^x\sin^2 x\mathrm{d}x$.(暨南大学 2005)　　(4) $\int x\mathrm{e}^x\cos x\mathrm{d}x$, $\int x\mathrm{e}^x\sin x\mathrm{d}x$.(西南大学 2007)

(5) $\int\frac{x\mathrm{e}^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x$.(地质大学 2007)　　(6) $\int\mathrm{e}^{\sin x}\frac{x\cos^3 x-\sin x}{\cos^2 x}\mathrm{d}x$.(南京理工 2006)

(7) $\int\frac{\ln\sin x}{\sin^2 x}\mathrm{d}x$.(太原科技 2005,苏州科技 2008,复旦大学 1997)

解题过程:

(1) 记 $I_1=\int\mathrm{e}^{ax}\cos bx\mathrm{d}x$, $I_2=\int\mathrm{e}^{ax}\sin bx\mathrm{d}x$,由分部积分法得

$$I_1=\frac{1}{a}\int\cos bx\mathrm{d}(\mathrm{e}^{ax})=\frac{1}{a}\left(\mathrm{e}^{ax}\cos bx+b\int\mathrm{e}^{ax}\sin b\mathrm{d}x\right)=\frac{1}{a}(\mathrm{e}^{ax}\cos bx+bI_2),$$

$$I_2=\frac{1}{a}\int\sin bx\mathrm{d}(\mathrm{e}^{ax})=\frac{1}{a}(\mathrm{e}^{ax}\sin bx-bI_1).$$

解得 $I_1=\int\mathrm{e}^{ax}\cos bx\mathrm{d}x=\mathrm{e}^{ax}\frac{b\sin bx+a\cos bx}{a^2+b^2}+C$, $I_2=\int\mathrm{e}^{ax}\sin bx\mathrm{d}x=\mathrm{e}^{ax}\frac{a\sin bx-b\cos bx}{a^2+b^2}+C$.

(2)记 $I=\int\cos(\ln x)\mathrm{d}x$， $J=\int\sin(\ln x)\mathrm{d}x$,由分部积分法得

$$I=x\cos(\ln x)+\int\sin(\ln x)\mathrm{d}x=x\cos(\ln x)+J,\ J=x\sin(\ln x)-\int\cos(\ln x)\mathrm{d}x=x\sin(\ln x)-I.$$

解得 $I=\int\cos(\ln x)\mathrm{d}x=\frac{1}{2}x(\cos(\ln x)+\sin(\ln x))+C$, $J=\int\sin(\ln x)\mathrm{d}x=\frac{1}{2}x(\sin(\ln x)-\cos(\ln x))+C$.

注:令 $t=\ln x$,则 $\mathrm{d}x=\mathrm{e}^t\mathrm{d}t$, $I=\int\mathrm{e}^t\cos t\mathrm{d}t$， $J=\int\mathrm{e}^t\sin t\mathrm{d}t$,本质上与(1)是同一种类型题.

(3) $\int e^x \sin^2 x dx = \frac{1}{2}\int e^x dx - \frac{1}{2}\int e^x \cos 2x dx = \frac{1}{2}e^x - \frac{1}{2}\int e^x \cos 2x dx$.

$$\int e^x \cos 2x dx = e^x \cos 2x + 2\int e^x \sin 2x dx = e^x(\cos 2x + 2\sin 2x) - 4\int e^x \cos 2x dx.$$

从而 $\int e^x \cos 2x dx = \frac{1}{5}e^x(\cos 2x + 2\sin 2x) + C$,

于是 $\int e^x \sin^2 x dx = \frac{1}{2}e^x - \frac{1}{10}e^x(\cos 2x + 2\sin 2x) + C$.

(4) $\int xe^x \cos x dx = \int x\cos x de^x = xe^x \cos x - \int e^x(\cos x - x\sin x)dx$

$$= xe^x \cos x - \int e^x \cos x dx + \int xe^x \sin x dx.$$

$$\int xe^x \sin x dx = \int x\sin x de^x = xe^x \sin x - \int e^x(\sin x + x\cos x)dx$$

$$= xe^x \sin x - \int e^x \sin x dx - \int xe^x \cos x dx.$$

由 $\int \cos x e^x dx = \frac{e^x(\sin x + \cos x)}{2}$, $\int \sin x e^x dx = \frac{e^x(\sin x - \cos x)}{2}$ 得

$$\int xe^x \cos x dx = \frac{xe^x(\sin x + \cos x)}{2} - \frac{e^x \cos x}{2} + C,$$

$$\int xe^x \sin x dx = -\frac{xe^x(\sin x - \cos x)}{2} + \frac{e^x \sin x}{2} + C.$$

(5) 令 $u = \arctan x$,则 $du = \frac{1}{1+x^2}dx$,则

$$\int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx = \int \frac{\tan u e^u}{\sec u}du = \int e^u \sin u du = \frac{1}{2}e^u(\sin u - \cos u) + C = \frac{x-1}{2\sqrt{1+x^2}}e^{\arctan x} + C.$$

(6) $\int e^{\sin x}\frac{x\cos^3 x - \sin x}{\cos^2 x}dx = \int x de^{\sin x} - \int e^{\sin x} d\left(\frac{1}{\cos x}\right) = e^{\sin x}\left(x - \frac{1}{\cos x}\right) + C$.

(7) $\int \frac{\ln\sin x}{\sin^2 x}dx = -\int \ln\sin x d(\cot x) = -\cot x \ln\sin x + \int \cot x \frac{\cos x}{\sin x}dx$

$$= -\cot x \ln\sin x + \int (\csc^2 x - 1)dx = -\cot x \ln\sin x - \cot^2 x - x + C.$$

10. 求下列积分.

(1) $\int e^{\sqrt{x}}dx$.(东华大学 2002,上海师大 2006,北京交大 2002,杭州师大 2012)

(2) $\int x^2 e^{-2x}dx$.(山东科技 2014,中科院武汉物理与数学研究所 2004)

(3) $\int e^{\sin x}\sin 2x dx$.(深圳大学 2009,山西大学 2004)

(4) $\int e^{2x}(\tan x + 1)^2 dx$.(上海财经 2005)

(5) $\int \frac{1-\ln x}{\ln^2 x}dx$.(中山大学 2009)

解题过程:

(1) 令 $u=\sqrt{x}$,则 $\int e^{\sqrt{x}}\,dx=\int e^{u}2u\,du=2(e^{u}u-e^{u})+C=2e^{\sqrt{x}}(\sqrt{x}-1)+C$.

(2) $\int x^2e^{-2x}dx=-\frac{1}{2}\int x^2d(e^{-2x})=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C$.

(3) $\int e^{\sin x}\sin 2x\,dx=2\int e^{\sin x}\sin x\cos x\,dx=2\int e^{\sin x}\sin x\,d(\sin x)=2\int\sin x\,d(e^{\sin x})$

$$=2\left(e^{\sin x}\sin x-\int e^{\sin x}d(\sin x)\right)=2(e^{\sin x}\sin x-e^{\sin x})+C=2e^{\sin x}(\sin x-1)+C.$$

(4) $\int e^{2x}(\tan x+1)^2dx=\int(\tan^2x+2\tan x+1)e^{2x}dx=\int e^{2x}\sec^2x\,dx+2\int e^{2x}\tan x\,dx$

$$=\int e^{2x}d\tan x+2\int e^{2x}\tan x\,dx=e^{2x}\tan x+C.$$

(5) 令 $t=\ln x$,则

$$\int\frac{1-\ln x}{\ln^2x}dx=\int\frac{1-t}{t^2}de^t=\int t^{-2}e^tdt-\int t^{-1}e^tdt=-\int e^tdt^{-1}-\int t^{-1}e^tdt$$

$$=-\left(e^tt^{-1}-\int t^{-1}e^tdt\right)-\int t^{-1}e^tdt=-e^tt^{-1}+C=-x(\ln x)^{-1}+C.$$

11. 求下列不定积分.

(1) $\int\frac{1+\ln x}{(x\ln x)^2}dx$.(湘潭大学 2012)

(2) $\int\frac{\ln x}{x\sqrt{1+\ln x}}dx$.(河北大学 2009)

(3) $\int x^x(1+\ln x)dx$.(山东科技 2008)

(4) $\int\frac{\ln(1+x)-\ln x}{x(x+1)}dx$.(西南大学 2011)

(5) $\int\left(\ln\ln x+\frac{1}{\ln x}\right)dx$.(南航 2012,华中师大 2006,河南大学 2002,暨南大学 2006,四川大学 2000)

解题过程:

(1) $\int\frac{1+\ln x}{(x\ln x)^2}dx=\int\frac{1}{(x\ln x)^2}d(x\ln x)=-\frac{1}{x\ln x}+C$.

(2) 令 $t=\ln x$, 则

$$\int\frac{\ln x}{x\sqrt{1+\ln x}}dx=\int\frac{t}{\sqrt{1+t}}dt=\int\left(\sqrt{1+t}-\frac{1}{\sqrt{1+t}}\right)dt=\frac{2}{3}(1+t)^{\frac{3}{2}}-2(1+t)^{\frac{1}{2}}+C$$

$$=\frac{2}{3}(1+\ln x)^{\frac{3}{2}}-2(1+\ln x)^{\frac{1}{2}}+C.$$

(3) $\int x^x(1+\ln x)dx=\int e^{x\ln x}(1+\ln x)dx=\int e^{x\ln x}d(x\ln x)=e^{x\ln x}+C$.

(4) 由于 $(\ln(1+x)-\ln x)'=-\frac{1}{x(x+1)}$,所以

$$\int\frac{\ln(1+x)-\ln x}{x(x+1)}dx=-\int(\ln(1+x)-\ln x)d(\ln(1+x)-\ln x)=-\frac{1}{2}[\ln(1+x)-\ln x]^2+C.$$

(5) $\int\left(\ln(\ln x)+\frac{1}{\ln x}\right)dx=\int\ln(\ln x)dx+\int\frac{1}{\ln x}dx$

$$= x\ln(\ln x) - \int x\cdot\frac{1}{x\ln x}dx + \int\frac{1}{\ln x}dx = x\ln(\ln x) + C.$$

12. 求下列不定积分.

(1) $\int\frac{\ln(1+x)}{x^2}dx$.(华东师大 2001,西安理工 2004,陕西师大 2005)

(2) $\int\frac{\ln(1+x^2)}{x^2}dx$.(暨南大学 2013)

(3) $\int\frac{\ln^3 x}{x^2}dx$.(湘潭大学 2011)

(4) $\int\frac{\ln^2 x}{x^3}dx$.(桂林电子科技 2007)

(5) $\int\frac{x}{1+x^2}\ln(1+x^2)dx$.(扬州大学 2003)

(6) $\int\frac{x\ln x}{(1+x^2)^2}dx$.(华南理工 2002)

解题过程:

(1) $\int\frac{\ln(1+x)}{x^2}dx = \int\ln(1+x)d\left(-\frac{1}{x}\right) = -\frac{1}{x}\ln(1+x) + \int\frac{dx}{x(1+x)} = -\frac{1}{x}\ln(1+x) + \ln\left|\frac{x}{1+x}\right| + C$.

(2) $\int\frac{\ln(1+x^2)}{x^2}dx = \int\ln(1+x^2)d\left(-\frac{1}{x}\right) = -\frac{1}{x}\ln(1+x^2) + 2\int\frac{dx}{1+x^2} = -\frac{1}{x}\ln(1+x^2) + 2\arctan x + C$.

(3) $\int\frac{\ln^3 x}{x^2}dx = -\frac{\ln^3 x}{x} + 3\int\frac{\ln^2 x}{x^2}dx = -\frac{\ln^3 x + 3\ln^2 x}{x} + 6\int\frac{\ln x}{x^2}dx$

$$= -\frac{\ln^3 x + 3\ln^2 x + 6\ln x}{x} + 6\int\frac{1}{x^2}dx = -\frac{\ln^3 x + 3\ln^2 x + 6\ln x + 6}{x} + C.$$

(4) $\int\frac{\ln^2 x}{x^3}dx = -\frac{1}{2}\int\ln^2 x d\left(\frac{1}{x^2}\right) = -\frac{1}{2}\frac{\ln^2 x}{x^2} + \frac{1}{2}\int\frac{1}{x^2}\cdot 2\ln x\cdot\frac{1}{x}dx$

$$= -\frac{1}{2}\frac{\ln^2 x}{x^2} - \frac{1}{2}\int\ln x d\left(\frac{1}{x^2}\right) = -\frac{\ln^2 x}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$$

(5) 令 $t = 1+x^2$,则

$$\int\frac{x}{1+x^2}\ln(1+x^2)dx = \frac{1}{2}\int\frac{1}{1+x^2}\ln(1+x^2)d(1+x^2) = \frac{1}{2}\int\frac{1}{t}\ln t dt = \frac{1}{4}\int d\ln^2 t$$

$$= \frac{1}{4}\ln^2 t + C = \frac{1}{4}\ln^2(1+x^2) + C.$$

(6) $\int\frac{x\ln x}{(1+x^2)^2}dx = -\frac{1}{2}\int\ln x d\left(\frac{1}{1+x^2}\right) = -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\int\frac{1}{x(1+x^2)}dx$

$$= -\frac{\ln x}{2(1+x^2)} + \frac{1}{2}\ln|x| - \frac{1}{4}\ln(1+x^2) + C.$$

13. 求下列不定积分.

(1) $\int\ln(\sqrt{x+2})dx$.(宁波大学 2012)

(2) $\int\ln(x+\sqrt{x^2-a^2})dx$.(华南理工 2003)

(3) $\int\ln(x+\sqrt{x^2+1})dx$.(燕山大学 2013)

(4) $\int\frac{x\ln(x+\sqrt{1+x^2})}{(1+x^2)^2}dx$.(武汉大学 2013)

(5) $\int x\ln\frac{1+x}{1-x}\mathrm{d}x$.(桂林电子科技 2013,湖北大学 2004,合肥工大 2002)

(6) $\int\sqrt{x}(\ln x)^2\mathrm{d}x$.(杭州师大 2013)

解题过程:

(1) 令 $t=\sqrt{x+2}$,则

$$\int\ln(\sqrt{x+2})\mathrm{d}x=\int 2t\ln t\mathrm{d}t=\int\ln t\mathrm{d}(t^2)=t^2\ln t-\frac{1}{2}t^2+C=(x+2)\left(\ln\sqrt{x+2}-\frac{1}{2}\right)+C .$$

(2) $\int\ln(x+\sqrt{x^2-a^2})\mathrm{d}x=x\ln(x+\sqrt{x^2-a^2})-\int\frac{x}{\sqrt{x^2-a^2}}\mathrm{d}x=x\ln(x+\sqrt{x^2-a^2})-\sqrt{x^2-a^2}+C$.

(3) $\int\ln(x+\sqrt{x^2+1})\mathrm{d}x=x\ln(x+\sqrt{x^2+1})-\int\frac{x}{\sqrt{x^2+1}}\mathrm{d}x=x\ln(x+\sqrt{x^2+1})-\sqrt{x^2+1}+C$.

(4) $$\int\frac{x\ln(x+\sqrt{1+x^2})}{(1+x^2)^2}\mathrm{d}x=-\frac{1}{2}\int\frac{\ln(x+\sqrt{1+x^2})}{(1+x^2)^2}\mathrm{d}\left(\frac{1}{1+x^2}\right)$$

$$=-\frac{1}{2}\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}+\frac{1}{2}\int\frac{1}{1+x^2}\cdot\frac{1}{\sqrt{1+x^2}}\mathrm{d}x.$$

令 $x=\tan\theta$,则

$$\int\frac{1}{1+x^2}\cdot\frac{1}{\sqrt{1+x^2}}\mathrm{d}x=\int\cos^3\theta\mathrm{d}(\tan\theta)=\int\cos\theta\mathrm{d}\theta=\sin\theta+C=\frac{x}{\sqrt{1+x^2}}+C .$$

于是
$$\int\frac{x\ln(x+\sqrt{1+x^2})}{(1+x^2)^2}\mathrm{d}x=-\frac{1}{2}\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}+\frac{1}{2}\frac{x}{\sqrt{1+x^2}}+C .$$

(5) $$\int x\ln\frac{1+x}{1-x}\mathrm{d}x=\frac{x^2}{2}\ln\frac{1+x}{1-x}-\int\frac{x^2}{2}\cdot\frac{2}{1-x^2}\mathrm{d}x=\frac{x^2}{2}\ln\frac{1+x}{1-x}+\int\frac{x^2-1+1}{x^2-1}\mathrm{d}x$$

$$=\frac{x^2}{2}\ln\frac{1+x}{1-x}+x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C .$$

(6) 令 $t=\sqrt{x}$,则

$$\int\sqrt{x}(\ln x)^2\mathrm{d}x=2\int t^2(\ln t^2)^2\mathrm{d}t=\frac{2}{3}\left[t^3(\ln t^2)^2-4\int t^2(\ln t^2)\mathrm{d}t\right]=\frac{2}{3}\left\{t^3(\ln t^2)^2-\frac{4}{3}\left[t^3(\ln t^2)-2\int t^2\mathrm{d}t\right]\right\}$$

$$=\frac{2}{3}\left\{t^3(\ln t^2)^2-\frac{4}{3}\left[t^3(\ln t^2)-\frac{4}{3}t^3\right]\right\}+C .$$

14. 求下列积分.

(1) $\int\frac{xe^{-x}}{(1-x)^2}\mathrm{d}x$.(华东师大 2004,漳州师院 2006)

(2) $\int\frac{xe^{x}}{(1+x)^2}\mathrm{d}x$.(广东工大 2014,湖南师大 2003,深圳大学 2004,青岛理工 2007,电子科技 2003,上海财经 2006)

(3) $\int \frac{x^2 e^x}{(x+2)^2}dx$.(云南大学 2001)

(4) $\int \frac{xe^x}{(1+e^x)^2}dx$.(上海师大 2003)

(5) $\int \frac{\ln x - 1}{(x+\ln x)^2}dx$.(电子科技 2005)

(6) $\int \frac{1+x}{x(1-xe^x)}dx$.(东南大学 2000)

(7) $\int \frac{1+x}{x(1+xe^x)}dx$.(燕山大学 2012)

(8) $\int \frac{1+x}{x(2+xe^x)}dx$.(山东科技 2013)

解题过程:

(1) $\int \frac{xe^{-x}}{(1-x)^2}dx = \int xe^{-x}d\left(\frac{1}{1-x}\right) = \frac{xe^{-x}}{1-x} - \int e^{-x}dx = \frac{xe^{-x}}{1-x} + e^{-x} + C$.

(2) $\int \frac{xe^x}{(1+x)^2}dx = -\int xe^x d\left(\frac{1}{1+x}\right) = -\frac{xe^x}{1+x} + \int \frac{1}{1+x}(e^x + xe^x)dx = \frac{e^x}{1+x} + C$.

(3) $\int \frac{x^2e^x}{(x+2)^2}dx = -\int x^2e^x d\left(\frac{1}{x+2}\right) = -\frac{x^2e^x}{x+2} + \int \frac{1}{x+2}(x^2e^x)'dx$

$$= -\frac{x^2e^x}{x+2} + \int xe^x dx = -\frac{x^2e^x}{2+x} + (x-1)e^x + C .$$

(4) $\int \frac{xe^x}{(1+e^x)^2}dx = \int xd\left(\frac{-1}{1+e^x}\right) = -\frac{x}{1+e^x} + \int \frac{1}{1+e^x}dx = -\frac{x}{1+e^x} + \int\left(1 - \frac{e^x}{1+e^x}\right)dx$

$$= -\frac{x}{1+e^x} + x - \ln(1+e^x) + C .$$

(5) 令 $\ln x = t$,则 $x = e^t, dx = e^t dt$.于是

$$\int \frac{\ln x - 1}{(x+\ln x)^2}dx = \int \frac{e^t(t-1)}{(e^t+t)^2}dt = -\int \frac{e^{-t}(t-1)}{(1+te^{-t})^2}dt = -\int \frac{1}{(1+te^{-t})^2}d(1+te^{-t})$$

$$= \frac{1}{1+te^{-t}} + C = \frac{x}{x+\ln x} + C .$$

(6) 令 $t = xe^x$,则

$$\int \frac{1+x}{x(1-xe^x)}dx = \int \frac{(1+x)e^x}{e^x x(1-xe^x)}dx = \int \frac{1}{e^x x(1-xe^x)}d(xe^x) = \int \frac{1}{t(1-t)}dt = \int \frac{1}{t}dt + \int \frac{1}{1-t}dt$$

$$= \ln|t| - \ln|1-t| + C = \ln\left|\frac{t}{1-t}\right| + C = \ln\left|\frac{xe^x}{1-xe^x}\right| + C .$$

(7) 令 $t = xe^x$,则

$$\int \frac{1+x}{x(1+xe^x)}dx = \int \frac{(1+x)e^x}{e^x x(1+xe^x)}dx = \int \frac{1}{e^x x(1+xe^x)}d(xe^x) = \int \frac{1}{t(1+t)}dt = \int \frac{1}{t}dt - \int \frac{1}{1+t}dt$$

$$= \ln|t| - \ln|1-t| + C = \ln\left|\frac{t}{1+t}\right| + C = \ln\left|\frac{xe^x}{1+xe^x}\right| + C .$$

(8) $\int \frac{1+x}{x(2+xe^x)}dx = \int \frac{(1+x)e^x}{e^x x(2+xe^x)}dx = \int \frac{1}{e^x x(2+xe^x)}d(xe^x) = \int \frac{1}{t(2+t)}dt = \frac{1}{2}\left(\int \frac{1}{t}dt - \int \frac{1}{2+t}dt\right)$

$$=\frac{1}{2}(\ln|t|-\ln|2+t|)+C=\frac{1}{2}\ln\left|\frac{t}{2+t}\right|+C=\frac{1}{2}\ln\left|\frac{xe^x}{2+xe^x}\right|+C.$$

15. 求下列积分.

(1) $\int\frac{e^x}{e^x+e^{-x}}dx$.(聊城大学 2011)

(2) $\int\frac{dx}{e^x+e^{-x}}$.(聊城大学 2008,青岛大学 2014,湘潭大学 2013)

(3) $\int\frac{1}{(e^x+1)^2}dx$.(山东科技 2009)

(4) $\int\frac{\ln(1+e^{-x})}{e^x+1}dx$.(苏州科技 2007,上海师大 2002)

(5) $\int\frac{dx}{\sqrt{1+e^{ax}}}$.($a$=1:桂林电子科技 2012,复旦大学 1998,a=2:上海大学 2013)

解题过程:

(1) $\int\frac{e^x}{e^x+e^{-x}}dx=\int\frac{e^{2x}}{e^{2x}+1}dx=\frac{1}{2}\int\frac{1}{e^{2x}+1}d(1+e^{2x})=\frac{1}{2}\ln(1+e^{2x})+C$.

(2) $\int\frac{dx}{e^x+e^{-x}}=\int\frac{d(e^x)}{e^{2x}+1}=\arctan e^x+C$.

(3) 令 $t=e^x$,则

$$\int\frac{dx}{(1+e^x)^2}=\int\frac{dt}{t(1+t)^2}=\int\left[\frac{1}{t(1+t)}-\frac{1}{(1+t)^2}\right]dt=\ln\left|\frac{t}{1+t}\right|+\frac{1}{1+t}+C=\ln\frac{e^x}{1+e^x}+\frac{1}{1+e^x}+C.$$

(4) 令 $t=1+e^{-x}$,则

$$\int\frac{\ln(1+e^{-x})}{e^x+1}dx=\int\frac{e^{-x}\ln(1+e^{-x})}{e^{-x}+1}dx=-\int\frac{\ln(1+e^{-x})}{e^{-x}+1}d(1+e^{-x})=-\int\frac{\ln t}{t}dt$$

$$=-\int\ln t\,d(\ln t)=-\frac{1}{2}\ln^2 t+C=-\frac{1}{2}\ln^2(1+e^{-x})+C.$$

(5) 令 $t=\sqrt{e^{ax}+1}$,则 $\int\frac{dx}{\sqrt{1+e^{ax}}}=\frac{1}{a}\int\frac{2tdt}{t(t^2-1)}=\frac{2}{a}\int\frac{dt}{t^2-1}=\frac{1}{a}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{1}{a}\ln\left|\frac{\sqrt{1+e^{ax}}-1}{\sqrt{1+e^{ax}}+1}\right|+C$.

16. 求下列积分.

(1) $\int e^x\left(\frac{1-x}{1+x}\right)^2dx$.(南京农大 2008)

(2) $\int e^x\frac{1+\sin x}{1+\cos x}dx$.(山东师大 2011,哈工大,华中科技)

(3) $\int e^{-x}\frac{1+\sin x}{1-\cos x}dx$.(武汉大学 2014)

(4) $\int\frac{xe^x dx}{\sqrt{e^x+1}}$.(上海师大 2004)