HEP World's Classics

Lectures on Cauchy's Problem in Linear Partial Differential Equations

线性偏微分方程中的柯西问题讲义

JACQUES HADAMARD

HIGHER EDUCATION PRESS

HEP World's Classics

Lectures on Cauchy's Problem in Linear Partial Differential Equations

线性偏微分方程中的柯西问题讲义

JACQUES HADAMARD

HIGHER EDUCATION PRESS

This edition first published 1923, Yale University Press

This book reproduces the text of the original edition. The content and language reflect the beliefs, practices and terminology of their time, and have not been updated.

图书在版编目（CIP）数据

线性偏微分方程中的柯西问题讲义 = Lectures on Cauchy's Problem in Linear Partial Differential Equations: 英文 / (法) 阿达马著. -- 北京: 高等教育出版社, 2016.2
ISBN 978-7-04-044804-7

Ⅰ. ①线… Ⅱ. ①阿… Ⅲ. ①线性方程 - 偏微分方程 - 初值问题 - 研究 - 英文 Ⅳ. ① O175

中国版本图书馆 CIP 数据核字（2016）第 019990 号

策划编辑 李华英　责任编辑 李华英　封面设计 王凌波　版式设计 于 婕
责任校对 刘 莉　责任印制 韩 刚

出版发行	高等教育出版社	咨询电话	400-810-0598
社　　址	北京市西城区德外大街 4 号	网　　址	http://www.hep.edu.cn
邮政编码	100120		http://www.hep.com.cn
印　　刷	涿州市星河印刷有限公司	网上订购	http://www.landraco.com
开　　本	787mm × 1092mm　1/16		http://www.landraco.com.cn
印　　张	18.75	版　　次	2016 年 2 月第 1 版
字　　数	490 千字	印　　次	2016 年 2 月第 1 次印刷
购书热线	010-58581118	定　　价	98.00 元

本书如有缺页、倒页、脱页等质量问题，请到所购图书销售部门联系调换

物 料 号　44804-00

Open and Read
Find Something Valuable

HEP World's Classics

There is a Chinese saying: "It is beneficial to open any book." It is even more fruitful to open and read classic books. The world is keeping on changing, but really fundamental and essential things stay the same since there is nothing new under the sun. Great ideas have been discovered and re-discovered, and they should be learnt and re-learnt. Classic books are our inheritance from all the previous generations and contain the best of knowledge and wisdom of all the people before us. They are timeless and universal. We cannot travel back in time, but we can converse with the originators of current theories through reading their books. Classic books have withstood the test of time. They are reliable and contain a wealth of original ideas. More importantly, they are also books which have not finished what they wanted or hoped to say. Consequently, they contain unearthed treasures and hidden seeds of new theories, which are waiting to be discovered. As it is often said: history is today. Proper understanding of the past work of giants is necessary to carry out properly the current and future researches and to make them to be a part of the history of science and mathematics. Reading classics books is not easy, but it is rewarding. Some modern interpretations and beautiful reformulations of the classics often miss the subtle and crucial points. Reading classics is also more than only accumulating knowledge, and the reader can learn from masters on how they asked questions, how they struggled to come up with new notions and theories to overcome problems, and answers to questions. Above all, probably the best reason to open classic books is the curiosity: what did people know, how did they express and communicate them, why did they do what they did? It can simply be fun!

This series of classic books by Higher Education Press contains a selection of best classic books in natural history, mathematics, physics, chemistry, information technology, geography, etc. from the past two thousand years. They contain masterpieces by the great people such Archimedes, Newton, Lavoisier, Dalton, Gauss, Darwin, Maxwell, and hence give a panorama of science and mathematics. They have been typeset in modern fonts for easier and more enjoyable reading. To help the reader understand difficult classics better, some volumes contain introductions and commentaries by experts. Though each classic book can stand in its own, reading them together will help the reader gain a bigger perspective of science and mathematics and understand better interconnection between seemingly unrelated topics and subjects.

Higher Education Press has been the largest publisher in China. Besides the long tradition of providing high quality books for proper education and training of university and graduate students, she has also set out to provide research monographs and references books to people at all levels around the world. Higher Education Press considers it her duty to keep the world science and mathematics community informed of what has been achieved in their subjects in easy and accessible formats. This series of classic books is an integral part of this effort.

LECTURES

ON

CAUCHY'S PROBLEM

IN

LINEAR PARTIAL DIFFERENTIAL EQUATIONS

BY

JACQUES HADAMARD, LL.D.

Member of the French Academy of Sciences

Foreign Honorary Member of the American Academy of Arts and Sciences

NEW HAVEN: YALE UNIVERSITY PRESS

LONDON: HUMPHREY MILFORD, OXFORD UNIVERSITY PRESS

MDCCCCXXIII

The title page of the original version

PREFACE

THE present volume is a résumé of my research work on the hyperbolic case in linear partial differential equations. I have had the happiness of speaking of some parts of it to an American audience at Columbia University (1911) and also had the honour of treating some points at the Universities of Rome (1916) and Zurich (1917)*. I am much indebted to Yale for having given me the opportunity to develop the whole of it, with the recent improvements which I have been able to make.

The origin of the following investigations is to be found in Riemann, Kirchhoff and still more Volterra's fundamental Memoirs on spherical and cylindrical waves. My endeavour has been to pursue the work of the Italian geometer, and so to improve and extend it that it may become applicable to all(normal) hyperbolic equations, instead of only to one of them. On the other hand, the present work may be considered as a continuation of my *Leçons sur la Propagation des Etudes et les Equations de l'Hydrodynamique*, and, even, as replacing several pages of the last chapter. The latter, indeed, was a first attempt, in which I only succeeded in showing the difficulties of the problem the solution of which I am now able to present.

Further extensions could also be given to such researches, including equations of higher orders, systems of equations, and even some applications to non-linear equations (the study of which has been undertaken in recent times, thanks to the theory of integral equations): which subjects, however, I have deliberately left aside, as the primary one constitutes a whole by itself. I shall be happy if some geometers succeed in extending the following methods to these new cases.

After Volterra's fundamental Memoirs of the *Acta Mathematica*, vol. XVIII, and his further contributions, we should have to mention, as developing and completing Volterra's point of view, the works of Tedone, Coulon and d'Adhémar†. The latter's volume *Les équations aux dérivées partielles à caractéristiques réelles* (Scientia Collection, Paris, Gauthier-Villars) includes a careful bibliographical re-

*I also mention a brief note read at the International Congress of Mathematicians at Strasbourg (September 1920).

†Picard's researches—which we shall quote in their place—are also essential in several parts of the present work. Such is also the case for Le Roux.

view, and another one has been given by Volterra himself in his Lectures delivered at Stockholm (published at Hermann's, Paris).We did not think it necessary to give a third one, even to add the mention of later works, and content ourselves with eventual quotations in footnotes, apologising in advance to the authors whom we may have forgotten*.

Reasons must also be given for the change of two terms which had been previously introduced and adopted in Science. One is "fundamental solution" replaced by "elementary solution"; the other consists in replacing the word "conormal", created by the finder (d'Adhémar) himself, by "transversal". The first has been done in order to avoid confusion with the "fundamental solutions" introduced by Poincaré and his successors (as solutions of homogeneous integral equations); the second for reasons of "economy of thought", as the notion in question already occurs in the Calculus of Variations, where it is denoted by the word "transversal".

I wish to express my heartiest thanks to two young American geometers, Mr Walsh and Mr Murray, whom I have been so pleased to see at Paris during the Academic year 1920—1921. They very kindly undertook to revise the English of the greater part of my manuscript. I fear many faults of language may have escaped detection, but that such errors are not more numerous is due to their useful and friendly help.

I am also greatly indebted to Prof. A. L. Underhill, of Minnesota, for his kind advices in correcting faults of language during the revision of proofs, and express to him my best thanks.

May 1923.

*Our own Memoirs on the subject have been inserted in the *Annales Scient. Ec. Norm. Sup.* (1904—1905) and the *Acta Mathematica* (vol. XXXI, 1908). We want to point out that the latter contains several errors in numerical coefficients, viz. in formula (30′), p. 349, where a denominator 2 must be cancelled (a factor 2 having similarly to be added in the preceding line), and in all formulæ relating to m even (corresponding to our Book IV), which must be corrected as in the present volume.

July 1921. J. H.

Several formulæ, being of general and constant use, have been denoted by special symbols, viz.:

	Stands for	*Introduced in*	
		Book	Section
(e_1)	Equation of vibrating strings.........	I	4
(C_1)	Corresponding Cauchy conditions......	I	4
(e_3), (C_3)	The same for equation of sound......	I	4
(e_2), (C_2)	The same for cylindrical waves......	I	$4a$
(E)	General form of the linear partial differential equation of the 2nd order.........	I	12
(**A**)	Partial differential equation for characteristics	I	13
(F_1)	Fundamental formula for Riemann's method	II	36
(g)	Green's formula............	II	40
(F)	Fundamental formula in general......	II	40
$(\mathcal{E})$	Adjoint equation............	II	41
(ϵ)	General linear equation in two variables...	II	42
(L_1), (L_2)	Differential equations for geodesics......	II	55

CONTENTS

BOOK I. GENERAL PROPERTIES OF CAUCHY'S PROBLEM

I. CAUCHY'S FUNDAMENTAL THEOREM CHARACTERISTICS 3
II. DISCUSSION OF CAUCHY'S RESULT 20

BOOK II. THE FUNDAMENTAL FORMULA AND THE ELEMENTARY SOLUTION

I. CLASSIC CASES AND RESULTS 41
II. THE FUNDAMENTAL FORMULA 51
III. THE ELEMENTARY SOLUTION 62
1. General Remarks 62
2. Solutions with an Algebroid Singularity 65
3. The Case of the Characteristic Conoid. The Elementary Solution 73
Additional Note on the Equations of Geodesics 97

BOOK III. THE EQUATIONS WITH AN ODD NUMBER OF INDEPENDENT VARIABLES

I. INTRODUCTION OF A NEW KIND OF IMPROPER INTEGRAL ... 103
1. Discussion of Preceding Results 103
2. The Finite Part of an Infinite Simple Integral 118
3. The Case of Multiple Integrals 125
4. Some Important Examples 134
II. THE INTEGRATION FOR AN ODD NUMBER OF INDEPENDENT VARIABLES 142
III. SYNTHESIS OF THE SOLUTION OBTAINED 161
IV. APPLICATIONS TO FAMILIAR EQUATIONS 184

BOOK IV. THE EQUATIONS WITH AN EVEN NUMBER OF INDEPENDENT VARIABLES AND THE METHOD OF DESCENT

I. INTEGRATION OF THE EQUATION IN $2m_1$ VARIABLES 193
1. General Formulæ 193
2. Familiar Examples 213
3. Application to a Discussion of Cauchy's Problem 222
II. OTHER APPLICATIONS OF THE PRINCIPLE OF DESCENT 236
1. Descent from m Even to m Odd 236
2. Properties of the Coefficients in the Elementary Solution 240
3. Treatment of Non-Analytic Equations 249

INDEX 279

BOOK I
GENERAL PROPERTIES OF CAUCHY'S PROBLEM

CHAPTER I

CAUCHY'S FUNDAMENTAL THEOREM
CHARACTERISTICS

WE shall have to deal with linear partial differential equations of the hyper- [3]
bolic type, and especially with Cauchy's problem concerning them.

What a linear partial differential equation is, is well known. What the hyperbolic type is, will be explained further on. Let us recall what Cauchy's problem is.

1. Boundary problems in general. A differential equation—whether ordinary or partial—admits of an infinite number of solutions. The older and classic point of view, concerning its integration, consisted in finding the so-called "general integral", i.e. a solution of the equation containing as many arbitrary elements (arbitrary parameters or arbitrary functions) as are necessary to represent any solution, save some exceptional ones.

But, in more recent research, especially as concerns partial differential equations, this point of view had to be given up, not only because of the difficulty or impossibility of obtaining this "general integral", but, above all, because the question does not by any means consist merely in its determination. The question, as set by most applications, does not consist in finding *any* solution u of the differential equation, but in choosing, amongst all those possible solutions, a particular one defined by properly given accessory conditions*. The partial differential equation ("indefinite equation" of some authors) has to be satisfied throughout the m-dimensional domain R (if we denote by m the number of independent variables) in which u shall exist; in other words, to be an identity, inasmuch as u is defined, and simultaneously the accessory conditions ("definite equations") have to be satisfied in points of the boundary of R. Examples of this will occur throughout these lectures.

If we have the general integral, there remains the question of choosing the [4]

*This even gives, as we conceive nowadays, the true manner of obtaining the general integral, as, by varying the accessory data in every possible way, we can, as a rule, get to any solution of our equation.

arbitrary elements in its expression so as to satisfy accessory conditions. In the case of ordinary differential equations, the arbitrary elements being numerical parameters, we have to determine them by an equal number of numerical equations, so that, at least theoretically, the question may be considered as solved, being reduced to ordinary algebra; but for partial differential equations, the arbitrary elements consist of functions, and the problem of their determination may be the chief difficulty in the question. For instance, we know the general integral of Laplace's equation $\nabla^2 u = 0$; but, nevertheless, this does not enable us to solve, without further and rather complicated calculations, the main problems depending on that equation, such as that of electric distribution.

The true questions which actually lie before us are, therefore, the "boundary problems", each of which consists in determining an unknown function u so as to satisfy:

(1) an "indefinite" partial differential equation;

(2) some "definite" boundary conditions.

Such a problem will be "correctly set" if those accessory conditions are such as to determine one and only one solution of the indefinite equation.

The simplest of boundary problems is Cauchy's problem.

2. Statement of Cauchy's problem. It represents, for partial differential equations, the exact analogue of the well-known fundamental problem in ordinary differential equations.

The theory of the latter was founded by Cauchy on the following theorem: Given an ordinary differential equation, say of the second order,

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \tag{1}$$

or, solving with respect to $\frac{d^2y}{dx^2}$,

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) = f(x, y, y'), \tag{1'}$$

a solution of this equation is (under proper hypotheses) determined if, for $x{=}0$,
[5] we know the numerical values y_0, y'_0 of y and $\frac{dy}{dx}$ $\left(\text{or, if the equation were of order } k, \text{ the numerical values of } y, \frac{dy}{dx}, \ldots, \frac{d^{k-1}y}{dx^{k-1}}\right)$.

Now let us start from a partial differential equation of the second order, such as (for two independent variables)

$$\phi\,(u, x, y, p, q, r, s, t) = 0 \tag{2}$$

or, if the number of independent variables is m,

$$\phi\,(u, x_i, p_i, r_i, s_{ik}) = 0, \tag{II}$$

where u is the unknown function; $x_1, x_2, \ldots, x_m$ the independent variables and $p_i (i = 1, 2, \ldots, m)$ stands for the first derivative $\frac{\partial u}{\partial x_i}$, r_i for the second derivative $\frac{\partial^2 u}{\partial x_i^2}$, s_{ik} for the second derivative $\frac{\partial^2 u}{\partial x_i \partial x_k}$. We especially deal with the linear case: that is, the left-hand side is linear with respect to u, p_i, r_i, s_{ik}, the coefficients being any given functions of $x_1, x_2, \ldots, x_m$. Now if we are asked to find a solution of that equation such that, for x_m=0, u and the first derivative $\frac{\partial u}{\partial x_m}$ be given functions of $x_1, x_2, \ldots, x_m$, viz.

$$u\left(x_1, x_2, \ldots, x_{m-1}, 0\right) = u_0\left(x_1, x_2, \ldots, x_{m-1}\right),$$
$$\frac{\partial u}{\partial x_m}\left(x_1, x_2, \ldots, x_{m-1}, 0\right) = u_1\left(x_1, x_2, \ldots, x_{m-1}\right),$$

this will be called *Cauchy's problem* with respect to $x_m = 0$; u_0 and u_1 will be *Cauchy's data* and x_m=0 the hypersurface* —here a hyperplane—which "bears" the data.

3. Of course, there is no reason to consider exclusively plane hypersurfaces. Let us imagine that the m-dimensional space be submitted to a point transforma- [6]
tion

$$\begin{cases} x_1 = G_1(X_1, \ldots, X_m), \\ x_2 = G_2(X_1, \ldots, X_m), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_m = G_m(X_1, \ldots, X_m) \end{cases} \tag{T}$$

(u not being altered by the transformation). The hyperplane x_m=0 will become, in that new X-space, a certain arbitrary surface S

$$G_m(X_1, \ldots, X_m) = 0. \tag{S}$$

Our differential equation being replaced by an analogous one

$$\Phi(u, X_1, X_2, \ldots, X_m, P_i, R_i, S_{ik}) = 0, \tag{IIa}$$

Cauchy's problem for that equation, with respect to the surface S, will consist in finding a solution of (IIa), satisfying, at every point of this surface, two conditions such as

$$u = u_0, \quad \frac{du}{dN} = U_1.$$

*In the m-dimensional space $(x_1, x_2, \ldots, x_m)$, we shall, for brevity's sake, call a *hypersurface* (or even a *surface*) the $(m-1)$-fold variety defined by one equation between the x's; we call an *edge* the $(m-2)$-fold variety defined by two equations. A *line* will, as usual, mean the locus of a point depending on one parameter; it will be a *straight* line if the x's are linear functions of the parameter.