FIFTY LECTURES FOR AMERICAN MATHEMATICS COMPETITIONS (VOLUME 6)

英文版

美国高中数学竞赛五十讲

第6卷

[美] 陈茧 [美] 陈三国 主编

$\sum_1^5 k^5 x_k = a^3$

$\sum_1^5 k^3 x_k = a^2$

$\sum_1^5 k x_k = a$

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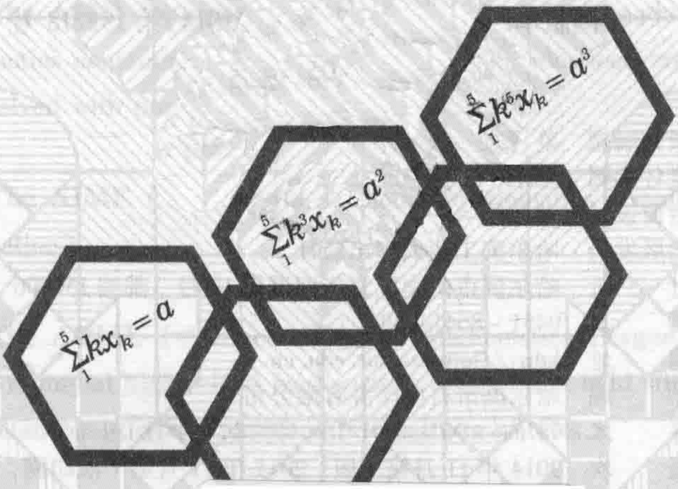

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内容简介

本书讲述了数学竞赛中常出现的知识点，还包括很多几何问题，每个知识点后配有大量的典型例题，书中的问题有趣，解题思路多样.

本书适合参加数学竞赛的高中生和教练员参考阅读，也适合数学能力很强的初中生及数学爱好者参考阅读.

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BASIC KNOWLEDGE

1. Two important terms

(1) Permutations

A permutation is an arrangement or a listing of objects in which the order is important. For example, if we have three numbers 1, 5, 9, there are 6 total permutations: {1,5,9}, {1,9,5}, {5,1,9}, {5,9,1}, {9,1,5}, {9,5,1}.

① There are n different elements, and we would like to arrange r of these elements with no repetition, where $1 \leqslant r \leqslant n$.

The number of such permutations is

$$P(n,r)=\frac{n!}{(n-r)!} \qquad (1)$$

② There are n different elements, and we would like to arrange all n of these elements with no repetition.

We let $r = n$ in (1) to get

$$P(n, n) = n! \qquad (2)$$

These n distinct objects can be permutated in $n!$ permutations.

The symbol ! (factorial) is defined as follows

$$0! = 1 \qquad (3)$$

and for integers $n \geq 1$

$$n! = n(n-1)\cdots 1 \qquad (4)$$

$$1! = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Proof of (2):

The first object can be chosen in n ways, the second object in $n-1$ ways, the third in $n-2$, etc. By the Fundamental Counting Principle, we have $n(n-1)(n-2)\cdots 2 \times 1 = n!$ ways.

Example 1: How many 5-digit positive integers can be formed by the digits of 0, 1, 2, 3 and 4?

Solution:

Since 12 340 is different from 13 240, the order in which the digits are arranged is important, so we will use permutations to solve this problem.

Method 1:

The number of permutations of 5 digits is $5! = 120$.

However, we also have to consider the permutations of 5 digits where 0 is the leftmost digit. Because 0 cannot be the leftmost digit, these permutations must be subtracted from the total permutations of 5 digits.

The number of permutations of 5 digits, with 0 as in the leftmost digit, is $4! = 24$. No positive 5-digit integers can be formed by these permutations, so the answer is then $120 - 24 = 96$.

Method 2:

The number of permutations with 0 as the units digit is $4! = 24$.

The number of permutations with 0 as the tens digit is $4! = 24$.

The number of permutations with 0 as the hundred digit is $4! = 24$.

The number of permutations with 0 as the thousand digit is $4! = 24$.

The answer is then $24 \times 4 = 96$.

(2) Combinations

Definition:

A combination is an arrangement or a listing of things in which order is not important.

Let n, r be nonnegative integers such that $0 \leqslant r \leqslant n$. The symbol $\binom{n}{r}$ (read "n choose m") is defined and denoted by

$$\binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!\,(n-r)!} \qquad (5)$$

Remember: $\binom{n}{0}=1$, $\binom{n}{1}=n$ and $\binom{n}{n}=1$.

Since $n-(n-r)=r$, we have

$$\binom{n}{r}=\binom{n}{n-r} \tag{6}$$

Unlike permutations, combinations are used when the order of the terms does not matter.

If we have n different elements, and it doesn't matter which order we arrange the elements, the number of combinations to arrange m elements where $1 \leqslant m \leqslant n$, is $\binom{n}{m}$.

Example 2: (a) In how many parts at most do n lines cut a plane?

(b) In how many parts at most do n planes cut a space?

Solution:

(a) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}$.

(b) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}$.

2. Two important rules

(1) The product rule (Fundamental Counting Principle) (Step Work)

When a task consists of k separate steps, if the first step can be done in n_1 ways, the second step can be done in n_2 ways, and so on through the kth step, which can be done in n_k ways, then the total number of possible results for completing the task is given by the product

$$N = n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k \qquad (7)$$

Example 3: (North Carolina Math Contest) There are 8 girls and 6 boys in the Math Club at Central High School. The Club needs to form a delegation to send to a conference, and the delegation must contain exactly two girls and two boys. The number of possible delegations that can be formed from the membership of the Club is ( )

(A) 48 (B) 420

(C) 576 (D) 1 680

(E) 2 304

Solution: (B).

In order to form the delegation, we must select two girls and two boys.

Step 1: Selecting two girls:

The number of combinations to select two girls out of eight is $\binom{8}{2}$.

Step 2: Selecting two boys:

The number of combinations to select two boys out of six is $\binom{6}{2}$.

By the product rule, we have the total number of distinct delegations

$$\binom{8}{2} \times \binom{6}{2} = 28 \times 15 = 420$$

(2) The sum rule (case work)

If an event E_1 can happen in n_1 ways, event E_2 can happen in n_2 ways, event E_k can happen in n_k ways, and if any event E_1, E_2, $\cdots$, E_k happens, the job is done, then the total ways to do the job is

$$N = n_1 + n_2 + \cdots + n_k \qquad (8)$$

Example 4: Hope High School has three elective courses for social studies and four electives for science. How many ways are there for Alex to select three electives from them this semester? There are ( ).

(A) 30 (B) 35

(C) 42 (D) 48

Solution: (B).

Case 1: Alex can select 1 social studies course and 2 science courses. By the product rule, there are

$$\binom{3}{1} \times \binom{4}{2} = 18$$

ways for Alex to select 1 social studies course and 2 science courses.

Case 2: Alex can select 2 social studies courses and 1 science course. By the product rule, there are

$$\binom{3}{2} \times \binom{4}{1} = 12$$

ways for Alex to select 2 social studies courses and 1 science course.

Case 3: Alex can select 3 social studies courses in $\binom{3}{3} = 1$ ways.

Case 4: Alex can select 3 science courses in $\binom{4}{3} = 4$

ways.

By the sum rule, there are $18 + 12 + 1 + 4 = 35$ total ways.

Example 5: How many two-digit numbers are there such that the units digit is greater than the tens digit?

Solution:

We want to count the number of two-digit number that have a larger units digit than the tens digit, giving us the following 4 cases:

Case 1: When the units digit is 9, the tens digit can be 1, 2, 3, 4, 5, 6, 7 or 8, so we have 8 such two-digit numbers.

Case 2: When the units digit is 8, the tens digit can be 1, 2, 3, 4, 5, 6 or 7, so we have 7 such two-digit numbers.

Case 3: When the units digit is 7, the tens digit can be 1, 2, 3, 4, 5 or 6, so we have 6 such two-digit numbers.

Case 4: When the units digit is 6, the tens digit can be 1, 2, 3, 4 or 5, so we have 5 such two-digit numbers.

Case 5: When the units digit is 5, the tens digit can be 1, 2, 3 or 4, so we have 4 such two-digit numbers.

Case 6: When the units digit is 4, the tens digit can be 1, 2 or 3, so we have 3 such two-digit numbers.

Case 7: When the units digit is 3, the tens digit can be 1 or 2, so we have 2 such two-digit numbers.

Case 8: When the units digit is 2, the tens digit can be 1, so we have 1 such two-digit number.

By the sum rule, we have

$$1+2+3+4+5+6+7+8=\frac{(1+8)\times 8}{2}=4\times 9=36$$

such two-digit numbers.

Example 6: (AMC) How many distinct four-digit numbers are divisible by 3 and have 23 as their last two digits? There are ( ).

(A) 27 (B) 30

(C) 33 (D) 81

(E) 90

Solution: (B).

Method 1: (official solution)

A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So a four-digit number $ab23$ is divisible by 3 if and only if the two-digit number ab leaves a remainder of 1 when divided by 3. There are 90 two-digit numbers, of which $\frac{90}{3}=30$ leave a remainder of 1 when divided by 3.

Method 2: (our solution)

A number is divisible by 3 if the sum of its digits are divisible by 3; therefore, the first two digits should have a sum of 1, 4, 7, 10, 13 or 16.

If the sum of the first two digits is 1, there is only 1 such number: 1 023.

If the sum of the first two digits is 4, there are 4 such numbers because

$$4=4+0=1+3=3+1=2+2$$

If the sum of the first two digits is 7, there are 7 such numbers because

$$7 = 7 + 0 = 6 + 1 = 1 + 6 = 5 + 2 = 2 + 5 = 4 + 3 = 3 + 4$$

If the sum of the first two digits is 10, there are 9 such numbers because

$$9 = 9 + 0 = 8 + 1 = 1 + 8 = 7 + 2 = 2 + 7 = 6 + 3$$
$$= 3 + 6 = 5 + 4 = 4 + 5$$

If the sum of the first two digits is 13, there are 6 such numbers because

$$13 = 9 + 4 = 4 + 9 = 8 + 5 = 5 + 8 = 7 + 6 = 6 + 7$$

If the sum of the first two digits is 16, there are 3 such numbers because

$$16 = 9 + 7 = 7 + 9 = 8 + 8$$

By the sum rule, we get the answer

$$1 + 4 + 7 + 9 + 6 + 3 = 30$$

3. Three important theorems

Theorem 1: (Grouping)

(a) Let the number of different objects be n. Divide n into r groups $A_1, A_2, \cdots, A_r$ such that there are n_1 objects in group A_1, n_2 objects in group $A_2, \cdots$, n_r objects in group A_r, where $n_1 + n_2 + \cdots + n_r = n$. The number of ways to do so is

$$N = \frac{n!}{n_1!\ n_2! \cdots n_r!} \qquad (9)$$

Proof:

There are $\binom{n}{n_1}$ ways to take out n_1 elements from n elements to form group A_1.

There are $\binom{n-n_1}{n_2}$ ways to take out n_2 elements from $n-n_1$ elements to form group A_2.

Continue the process until there are n_r elements left to form group A_r.

The total number of ways, based on the Fundamental Counting Principle, is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n_r}{n_r}=\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

(b) Let there be r types of objects: n_1 of type 1, n_2 of type 2, etc. The number of ways in which these $n_1+n_2+\cdots+n_r=n$ objects can be rearranged is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!} \qquad (10)$$

Example 7: A gardener plants eight trees out of three maple trees, two oak trees, and four birch trees in a row. How many ways are there?

Solution:

Case 1: With two maple trees, two oak trees, and four birch trees, by (9), we have $\frac{8!}{2!\,2!\,4!}=420$ ways.

Case 2: With three maple trees, one oak tree, and four birch trees, there are $\frac{8!}{3!\,1!\,4!}=280$ ways.

Case 3: With three maple trees, two oak trees, and three birch trees, there are $\frac{8!}{3!\,2!\,3!}=560$ ways.

By the sum rule, we know that the total number of ways to plant the trees is

$$420+280+560=1\ 260$$

Example 8: Five numbers 1, 2, 3, 4 and 5 are arranged in a row like $a_1a_2a_3a_4a_5$. How many arrangements are there such that $a_1\neq 1$, $a_2\neq 2$, $a_3\neq 3$, $a_4\neq 4$, $a_5\neq 5$?

Solution:

We let 2 be the leftmost number. As shown in Fig. 1, there are 11 such arrangements:

Since the leftmost digit can also be 3, 4 or 5, so the answer by the product rule will be $11\times 4=44$.

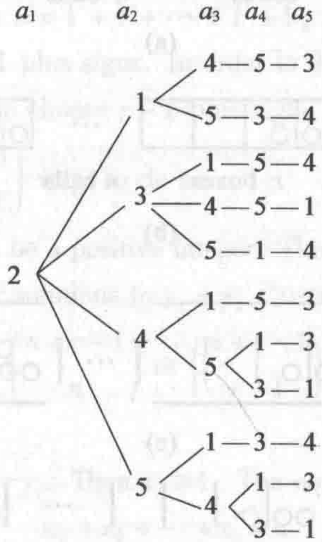

Fig. 1

Theorem 2: (**Combinations with Repetitions**)

(a) n identical balls are put into r labeled boxes and the number of balls in each box is not limited. The number of ways is

$$\binom{n+r-1}{n} \text{ or } \binom{n+r-1}{r-1} \tag{11}$$

Proof:

Put r labeled boxes next to each other as shown in Fig. 2. Put n balls into these boxes. Next, we line these boxes up next to each other (Fig. 2(b)). Now we take apart the top and bottom sides of the each box and the two sides of the two boxes at the end (Fig. 2(c)) , resulting Fig. 2(d).

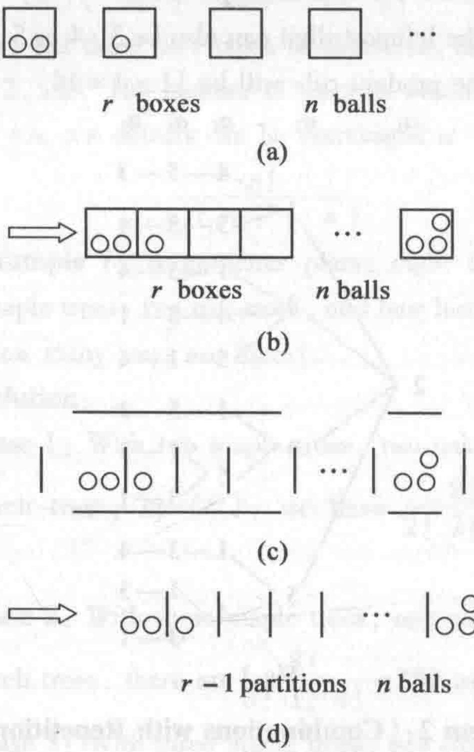

Fig. 2

The problem now becomes finding the number of ways to permute n identical balls with $r-1$ identical partitions: $\dfrac{(n+r-1)!}{n!\,(r-1)!}$ or $\binom{n+r-1}{n}$ or $\binom{n+r-1}{r-1}$.

(b) The number of terms in the expansion of $(x_1 + x_2 + x_3 + \cdots + x_r)^n$, after the like terms combined, is

$$\binom{n+r-1}{n} \text{ or } \binom{n+r-1}{r-1} \qquad (12)$$

(c) Let n be a positive integer. The number of positive integer solutions to $x_1 + x_2 + \cdots + x_r = n$ is

$$\binom{n-1}{r-1} \qquad (13)$$

Proof:

Write n as $n = 1 + 1 + \cdots + 1 + 1$, where there are n 1's and $n-1$ plus signs. In order to decompose n into r summands, we choose $r-1$ plus signs from the $n-1$, giving us $\binom{n-1}{r-1}$ ways to do so.

(d) Let n be a positive integer. The number of non-negative integer solutions to $y_1 + y_2 + \cdots + y_r = n$ is

$$\binom{n+r-1}{n} \text{ or } \binom{n+r-1}{r-1} \qquad (14)$$

Proof:

Set $x_r - 1 = y_r$. Then $x_r \geqslant 1$. The equation

$$x_1 + x_2 + \cdots + x_r = n$$

is equivalent to $x_1 + x_2 + \cdots + x_r = n + r$, which has $\binom{n+r-1}{r-1}$ solutions.

Example 9: A baking company produces four different cookies: Chocolate Chip Cookies, Peanut Butter