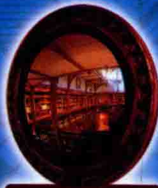
FIFTY LECTURES FOR AMERICAN MATHEMATICS COMPETITIONS (VOLUME 1)



英文版

美国高中数学竞赛五十讲

第1卷

[美] 陈茧　[美] 陈三国　主编

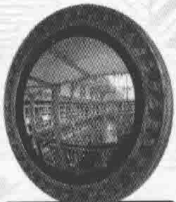
$\sum_1^5 k^5 x_k = a^3$

$\sum_1^5 k^3 x_k = a^2$

$\sum_1^5 k x_k = a$

哈爾濱工業大學出版社
HARBIN INSTITUTE OF TECHNOLOGY PRESS

FIFTY LECTURES FOR AMERICAN MATHEMATICS COMPETITIONS (VOLUME 1)



英文版

美国高中数学竞赛五十讲

第1卷

[美] 陈茧 [美] 陈三国 主编

$\sum_1^5 k^5 x_k = a^3$

$\sum_1^5 k^3 x_k = a^2$

$\sum_1^5 k x_k = a$

哈爾濱工業大學出版社
HARBIN INSTITUTE OF TECHNOLOGY PRESS

内容简介

本书讲述了数学竞赛中常出现的知识点，还包括很多几何问题，每个知识点后配有大量的典型例题，书中的问题有趣，解题思路多样.

本书适合参加数学竞赛的高中生和教练员参考阅读，也适合数学很强的初中生及数学爱好者参考阅读.

图书在版编目(CIP)数据

美国高中数学竞赛五十讲. 第1卷：英文/(美)陈茧，(美)陈三国主编. —哈尔滨：哈尔滨工业大学出版社，2014.8

ISBN 978 -7 -5603 -4794 -3

Ⅰ. ①美… Ⅱ. ①陈… ②陈… Ⅲ. ①中学数学课 - 竞赛题 - 题解 - 英文 Ⅳ. ①G634.605

中国版本图书馆 CIP 数据核字(2014)第131681号

策划编辑　刘培杰　张永芹
责任编辑　张永芹　邵长玲
封面设计　孙茵艾
出版发行　哈尔滨工业大学出版社
社　　址　哈尔滨市南岗区复华四道街10号　邮编150006
传　　真　0451 -86414749
网　　址　http://hitpress.hit.edu.cn
印　　刷　哈尔滨市石桥印务有限公司
开　　本　787mm×960mm　1/16　印张10.75　字数120千字
版　　次　2014年8月第1版　2014年8月第1次印刷
书　　号　ISBN 978 -7 -5603 -4794 -3
定　　价　28.00元

目录

Algebraic Manipulation

Chapter 1

BASIC KNOWLEDGE

Below is a list of useful equations to be aware of and know. They can all be derived through expanding or factoring.

Perfect square trinomial

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^2 = (x-y)^2 + 4xy$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x-y)^2 = (x+y)^2 - 4xy$$

$$(x+y)^2 + (x-y)^2 = 2(x^2+y^2)$$

$$(x+y)^2 - (x-y)^2 = 4xy$$

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}[(x-y)^2 + (y-z)^2 + (z-x)^2]$$

$$(x+y+z+w)^2 = x^2+y^2+z^2+w^2+2xy+2xz+2xw+2yz+2yw+2zw$$

Difference and sum of two squares

$$x^2+y^2=(x+y)^2-2xy$$
$$x^2+y^2=(x-y)^2+2xy$$
$$x^2-y^2=(x-y)(x+y)$$

Difference and sum of two cubes

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$
$$x^3-y^3=(x-y)(x^2+xy+y^2)$$
$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3 = x^3+y^3+3xy(x+y)$$
$$(x-y)^3 = x^3-3x^2y+3xy^2-y^3 = x^3-y^3-3xy(x-y)$$
$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\cdots+y^{n-1})$$

for all n

$$x^n-y^n=(x+y)(x^{n-1}-x^{n-2}y+\cdots-y^{n-1})$$

for all even n

$$x^n+y^n=(x+y)(x^{n-1}-x^{n-2}y+\cdots+y^{n-1})$$

for all odd n.

EXAMPLES

Example 1: Find $m^3+\frac{1}{m^3}$ if $m+\frac{1}{m}=2$.

Solution: 2.

Method 1: Multiplying both sides of $m+\frac{1}{m}=2$ by

m yields

$$m^2+1=2m \Rightarrow m^2-2m+1=0 \Rightarrow (m-1)^2=0$$

Solving this equation, we get $m=1$.

Therefore

$$m^3+\frac{1}{m^3}=1^3+\frac{1}{1^3}=1+1=2$$

Method 2: We know that

$$\begin{aligned}(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\ &= x^3+y^3+3xy(x+y)\end{aligned}$$

So

$$x^3+y^3=(x+y)^3-3xy(x+y) \tag{1}$$

Substituting in $x=m$ and $y=\frac{1}{m}$ into (1) gives us

$$\begin{aligned}m^3+\frac{1}{m^3} &= (m+\frac{1}{m})^3-3m\cdot\frac{1}{m}(m+\frac{1}{m})\\ &= 2^3-3\times2=2\end{aligned}$$

Method 3: We know that

$$x^3+y^3=(x+y)(x^2-xy+y^2)$$

Substituting $x=m$ and $y=\frac{1}{m}$ into the above equation, we get

$$\begin{aligned}m^3+\frac{1}{m^3} &= (m+\frac{1}{m})(m^2-m\cdot\frac{1}{m}+\frac{1}{m^2})\\ &= 2[(m+\frac{1}{m})^2-3]=2(2^2-3)=2\end{aligned}$$

Method 4: Multiplying both sides of $m+\frac{1}{m}=2$ by $(m^2+\frac{1}{m^2}-1)$ yields

$$(m^2+\frac{1}{m^2}-1)(m+\frac{1}{m})=2(m^2+\frac{1}{m^2}-1)$$

$$\Rightarrow m^3+\frac{1}{m^3}$$

$$=2[(m+\frac{1}{m})^2-2-1]$$

$$=2$$

Example 2: Find x^2+y^2 if (x,y) is a solution to the system of equation

$$xy=4$$

$$x^2y+xy^2+x+y=45$$

Solution: 73.

Factor

$$x^2y+xy^2+x+y=xy(x+y)+(x+y)$$

$$=(4+1)(x+y)$$

We are given that the expression above equals 45, so

$$(4+1)(x+y)=45\Rightarrow x+y=9$$

We know that $x^2+y^2=(x+y)^2-2xy$. Substituting in 4 for xy and 9^2 for $(x+y)^2$ into this equation, we get

$$x^2+y^2=81-8=73$$

Example 3: Find $m^4+\frac{1}{m^4}$ if $m+\frac{1}{m}=4$.

Solution: 194.

Substituting m^2 for x and $\frac{1}{m^2}$ for y into $x^2+y^2=(x+y)^2-2xy$ gives us

$$m^4+\frac{1}{m^4}=\left(m^2+\frac{1}{m^2}\right)^2-2=\left[\left(m+\frac{1}{m}\right)^2-2\right]^2-2$$

$$= (16 - 2)^2 - 2 = 194$$

Example 4: Find $m^8 + \frac{1}{m^8}$ if $m - \frac{1}{m} = 1$.

Solution: 47.

Substituting m^4 as x and $\frac{1}{m^4}$ as y into $x^2 + y^2 = (x + y)^2 - 2xy$ gives us

$$m^4 + \frac{1}{m^4} = \left(m^2 + \frac{1}{m^2}\right)^2 - 2 = \left[\left(m - \frac{1}{m}\right)^2 + 2\right]^2 - 2$$
$$= 9 - 2 = 7$$

Substituting m^4 as x and $\frac{1}{m^4}$ as y into $x^2 + y^2 = (x + y)^2 - 2xy$ yields

$$m^8 + \frac{1}{m^8} = \left(m^4 + \frac{1}{m^4}\right)^2 - 2 = 49 - 2 = 47$$

Example 5: Find $m^3 - \frac{1}{m^3}$ if $y = \sqrt{m^2 - m - 1} + \sqrt{m + 1 - m^2}$, where both m and y are real numbers.

Solution: 4.

Since y is given to be a real number, the expressions under the square roots must be greater than or equal to 0. Therefore

$$\begin{cases} m^2 - m - 1 \geqslant 0 \\ m + 1 - m^2 \geqslant 0 \Rightarrow m^2 - m - 1 \leqslant 0 \end{cases}$$

Since $m^2 - m - 1$ must be both greater than or equal to 0 and less than or equal to 0

$$m^2 - m - 1 = 0 \Rightarrow m - \frac{1}{m} = 1 \quad (m \neq 0)$$

$$m^3 - \frac{1}{m^3} = \left(m - \frac{1}{m}\right)\left(m^2 + \frac{1}{m^2} + 1\right)$$
$$= \left(m - \frac{1}{m}\right)\left[\left(m - \frac{1}{m}\right)^2 + 3\right] = 4$$

Example 6: Find $\frac{2x+3xy-2y}{x-2xy-y}$ if $\frac{1}{x} - \frac{1}{y} = 3$.

Solution: $\frac{3}{5}$.

Method 1: Since the denominator of a fraction cannot be 0, we know that $x \neq 0$, $y \neq 0$ and $x - 2xy - y \neq 0$.

Dividing each term of $\frac{2x+3xy-2y}{x-2xy-y}$ by xy yields

$$\frac{\frac{2}{y} - \frac{2}{x} + 3}{\frac{1}{y} - \frac{1}{x} - 2} = \frac{-2\left(\frac{1}{x} - \frac{1}{y}\right) + 3}{-\left(\frac{1}{x} - \frac{1}{y}\right) - 2}$$
$$= \frac{-2 \times 3 + 3}{-3 - 2} = \frac{3}{5}$$

Method 2: Multiplying both sides of $\frac{1}{x} - \frac{1}{y} = 3$ by xy gives us $y - x = 3xy$.

Hence

$$\frac{2x+3xy-2y}{x-2xy-y} = \frac{-2(y-x)+3xy}{-(y-x)-2xy}$$
$$= \frac{-6xy+3xy}{-3xy-2xy} = \frac{3}{5}$$

Method 3: Solving for x in the equation $\frac{1}{x} - \frac{1}{y} = 3$ gives us

$$x = \frac{y}{3y+1}$$

Hence

$$\frac{2x+3xy-2y}{x-2xy-y}=\frac{\frac{2y}{3y+1}+\frac{3y^2}{3y+1}-\frac{6y^2+2y}{3y+1}}{\frac{y}{3y+1}-\frac{2y^2}{3y+1}-\frac{3y^2+y}{3y+1}}$$

$$=\frac{2y+3y^2-6y^2-2y}{y-2y^2-3y^2-y}$$

$$=\frac{-3y^2}{-5y^2}=\frac{3}{5}$$

Method 4: Let

$$\frac{2x+3xy-2y}{x-2xy-y}=k$$

$$2x+3xy-2y=k(x-2xy-y)$$
$$\Rightarrow 2x+3xy-2y=kx-2kxy-ky$$
$$\Rightarrow 2y-2x-ky+kx=3xy+2kxy$$
$$\Rightarrow (2-k)(y-x)=(3+2k)xy \qquad (2)$$

Since $\frac{1}{x}-\frac{1}{y}=3$, multiplying both sides by xy yields

$$y-x=3xy \qquad (3)$$

Dividing equation (3) by equation (2) gives us

(2) ÷ (3)

$$2-k=\frac{3+2k}{3}\Rightarrow k=\frac{3}{5}$$

$$k=\frac{2x+3xy-2y}{x-2xy-y}=\frac{3}{5}$$

Example 7: Find $\frac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}$ if $\sqrt{x}(\sqrt{x}+\sqrt{y})=3\sqrt{y}(\sqrt{x}+5\sqrt{y})$. x and y are positive numbers.

Solution: 2.

We are given that

$$\sqrt{x}(\sqrt{x}+\sqrt{y})=3\sqrt{y}(\sqrt{x}+5\sqrt{y})$$

Expanding both sides gives us

$$x+\sqrt{xy}=3\sqrt{xy}+15y \Rightarrow x-2\sqrt{xy}-15y=0$$

Factoring gives us

$$(\sqrt{x}-5\sqrt{y})(\sqrt{x}+3\sqrt{y})=0$$

Since $\sqrt{x}+3\sqrt{y}\neq 0$, because $x>0, y>0$, $(\sqrt{x}-5\sqrt{y})\cdot(\sqrt{x}+3\sqrt{y})$ can only equal 0 if and only if

$$\sqrt{x}-5\sqrt{y}=0 \Rightarrow x=25y$$

Substituting in $25y$ as x into $\frac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}$ gives us

$$\frac{2x+\sqrt{xy}+3y}{x+\sqrt{xy}-y}=\frac{2\times 25y+5y+3y}{25y+5y-y}=\frac{58}{29}=2$$

Example 8: Find the value of $\frac{3}{2}x^2-x+1$ if the value of $3x^2-2x+6$ is 8.

Solution: 2.

Method 1:

Since

$$3x^2-2x+6=8,\ 3x^2-2x=2$$

$$\frac{3}{2}x^2-x+1=\frac{1}{2}(3x^2-2x)+1=\frac{1}{2}\times 2+1=2$$

Method 2:

We are given that $3x^2-2x+6=8$

or

$$3x^2-2x+2=4 \qquad (4)$$

Dividing both sides of (4) by 2 gives us the answer

$$\frac{3}{2}x^2 - x + 1 = 2$$

Example 9: Calculate$(2+1)\cdot(2^2+1)\cdot(2^4+1)\cdot\cdots\cdot(2^{32}+1)+1$.

Solution: 2^{64}.

Notice that

$$(2-1)(2+1) = 2^2 - 1$$

We multiply the given expression by $1 = 2 - 1$

$$[(2-1)(2+1)](2^2+1)(2^4+1)\cdots(2^{32}+1)+1$$
$$=[(2^2-1)(2^2+1)](2^4+1)\cdots(2^{32}+1)+1$$
$$=[(2^8-1)(2^8+1)(2^{16}+1)\cdots(2^{32}+1)]+1$$
$$=(2^{32}-1)(2^{32}+1)+1 = 2^{64}-1+1 = 2^{64}$$

Example 10: (AMC) The expression $x^2 - y^2 - z^2 + 2yz + x + y - z$ has(   ).

(A) no linear factor with integer coefficients and integer exponents

(B) the factor $-x+y+z$

(C) the factor $x-y-z+1$

(D) the factor $x+y-z+1$

(E) the factor $x-y+z+1$

Solution: (E).

Method 1: (official solution)

$$x^2 - y^2 - z^2 + 2yz + x + y - z$$
$$= x^2 - (y^2 - 2yz + z^2) + x + y - z$$
$$= x^2 - (y-z)^2 + x + y - z$$
$$= (x+y-z)(x-y+z) + x + y - z$$
$$= (x+y-z)(x-y+z+1)$$

Method 2: (our solution)

Let

$$x^2 - y^2 - z^2 + 2yz + x + y - z = 0$$

Rearrange the terms to give us the quadratic

$$y^2 - (2z + 1)y + z + z^2 - x^2 - x = 0$$

Applying the quadratic formula, we get the roots

$$y_{1,2} = \frac{(2z+1) \pm \sqrt{[-(2z+1)]^2 - 4 \times 1 \times (z + z^2 - x^2 - x)}}{2}$$

$$= \frac{(2z+1) \pm \sqrt{(2x+1)^2}}{2} = \frac{2z + 1 \pm 2x + 1}{2}$$

$$y_1 = \frac{2z + 1 + 2x + 1}{2} = z + x + 1$$

and

$$y_2 = \frac{2z + 1 - 2x - 1}{2} = z - x$$

Therefore, we have

$$x^2 - y^2 - z^2 + 2yz + x + y - z$$

$$= (y - (z + x + 1))(y - (z - x))$$

$$= (x + y - z)(x - y + z + 1)$$

Example 11: Both a and b are real numbers with

$$\left(m^3 + \frac{1}{m^3} - a\right)^2 + \left(m + \frac{1}{m} - b\right)^2 = 0$$

Prove: $b(b^2 - 3) = a$.

Solution: Since the square of a number is always greater than or equal to 0, in order for the sum of two squares to be 0

$$m^3 + \frac{1}{m^3} - a = 0,\ m + \frac{1}{m} - b = 0$$

$$m^3+\frac{1}{m^3}=a,\ m+\frac{1}{m}=b$$

So

$$a=m^3+\frac{1}{m^3}=\left(m+\frac{1}{m}\right)\left(m^2-m\left(\frac{1}{m}\right)+\frac{1}{m^2}\right)$$

$$=\left(m+\frac{1}{m}\right)\left[\left(m+\frac{1}{m}\right)^2-3\right]=b(b^2-3)$$

Example 12: Find the greatest positive integer not exceeding $(\sqrt{7}+\sqrt{3})^6$.

Solution: 7 039.

Let

$$x=\sqrt{7}+\sqrt{3},\ y=\sqrt{7}-\sqrt{3}$$

$$x+y=(\sqrt{7}+\sqrt{3})+(\sqrt{7}-\sqrt{3})=2\sqrt{7}$$

$$xy=(\sqrt{7}+\sqrt{3})(\sqrt{7}-\sqrt{3})=7-3=4$$

$$x^2+y^2=(x+y)^2-2xy=20$$

$$x^6+y^6=(x^2+y^2)^3-3(x^2+y^2)\cdot x^2\cdot y^2=7\ 040$$

$$(\sqrt{7}+\sqrt{3})^6+(\sqrt{7}-\sqrt{3})^6=7\ 040$$

$$(\sqrt{7}+\sqrt{3})^6=7\ 040-(\sqrt{7}-\sqrt{3})^6$$

We know that $0<\sqrt{7}-\sqrt{3}<1$, so $0<(\sqrt{7}-\sqrt{3})^6<1$.

The greatest positive integer not exceeding $(\sqrt{7}+\sqrt{3})^6$ is $7\ 040-1=7\ 039$.

Example 13: Determine the largest real number z such that

$$x+y+z=5$$

$$xy+yz+xz=3$$

and x, y are also real.

Solution: $\frac{13}{3}$.

Since

$$x+y+z=5$$

we have

$$x+y=5-z$$

Squaring both sides gives us $(x+y)^2=(5-z)^2$.

Since $xy+yz+xz=3$, we have

$$xy=3-xz-yz=3-z(x+y)=3-z(5-z)$$

$$\begin{aligned}(x-y)^2 &= (x+y)^2-4xy \\ &= (5-z)^2-4[3-z(5-z)] \\ &= -3z^2+10z+13=(13-3z)(1+z)\end{aligned}$$

Since $(x-y)^2 \geqslant 0$, this means that

$$(13-3z)(1+z) \geqslant 0$$

Solving the inequality gives us

$$-1 \leqslant z \leqslant \frac{13}{3}$$

The largest real number z is $\frac{13}{3}$ when $x=y=\frac{1}{3}$.

Example 14: What is the value of m such that x^2-x-1 is a factor of mx^7+nx^6+1? m and n are integers.

Solution: 8.

Using the quadratic formula, we solve the quadratic equation $x^2-x-1=0$ to get the roots

$$x_1=\frac{1+\sqrt{5}}{2},\ x_2=\frac{1-\sqrt{5}}{2}$$

We can observe that $x_1+x_2=1$ and $x_1 x_2=-1$.

Thus, $(x_1+x_2)^2=x_1^2+x_2^2+2x_1x_2=1$ and

$$x_1^2+x_2^2=3 \quad (5)$$

We know that $x^2 - x - 1$ is a factor of $mx^7 + nx^6 + 1$, x_1 and x_2 are also the roots of $mx^7 + nx^6 + 1 = 0$.

Therefore we have

$$mx_1{}^7 + nx_1{}^6 = -1 \tag{6}$$

and

$$mx_2{}^7 + nx_2{}^6 = -1 \tag{7}$$

$$(6) \times x_2{}^6 \Rightarrow m\,x_1{}^7\,x_2{}^6 + nx_1{}^6\,x_2{}^6 = -x_2{}^6$$

$$\Rightarrow mx_1(-1)^6 + n(-1)^6 = -x_2{}^6$$

$$\Rightarrow mx_1 + n = -x_2{}^6 \tag{8}$$

$(7) \times x_1{}^6$

$$mx_2 + n = -x_1{}^6 \tag{9}$$

$(8) - (9)$

$$m(x_1 - x_2) = x_1{}^6 - x_2{}^6$$

So

$$m = \frac{x_1^6 - x_2^6}{x_1 - x_2} = \frac{(x_1^3 - x_2^3)(x_1^3 + x_2^3)}{x_1 - x_2}$$

$$= \frac{(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)(x_1^3 + x_2^3)}{x_1 - x_2}$$

$$= (x_1^2 + x_1x_2 + x_2^2)(x_1^3 + x_2^3)$$

$$= (x_1^2 + x_1x_2 + x_2^2)(x_1 + x_2)(x_1^2 - x_1x_2 + x_2^2)$$

$$= (3-1)(1)(3+1) = 8$$

PROBLEMS

Problem 1: Find $m^2 + \frac{1}{m^2}$ if $m + \frac{1}{m} = 4$.

Problem 2: Find $a^3 + \frac{1}{a^3}$ if $a + \frac{1}{a} = \sqrt{3}$. It