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Spatial Control of Laser Light in Atomic Vapors and Dielectric Media

(激光在原子气体及电介质中的空间控制)

Zhang Yiqi   Belić Milivoj   Zhang Yanpeng
（张贻齐   米利沃·贝里奇   张彦鹏）

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In this book, the authors introduce their achievements in spatial control of light in atomic vapors and dielectric media. There are five chapters in this book. In Chapter 1, the basic concepts and theories used in this book are introduced. From Chapter 2 to Chapter 4, the authors report their research results in detail. The topics include photonic topological insulators, Talbot effect, optical rogue waves, optical vortices, azimuthons, incoherent solitons, Airy beams, Bessel beams, Fresnel diffraction, and fractional Schrödinger equations, which are optical hot subjects in recent years. The authors summarize the book in Chapter 5, and meanwhile make an outlook on their future work. Whilst all the chapters are seemingly independent in form, they connect with each other in content.

This book can be a reference for researchers as well as graduate students in optical physics. In addition, this book is also good and helpful to undergraduates majored in physics and opto-electronics.

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《博士后文库》序言

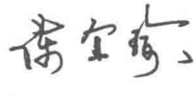
博士后制度已有一百多年的历史。世界上普遍认为，博士后研究经历不仅是博士们在取得博士学位后找到理想工作前的过渡阶段，而且也被看成是未来科学家职业生涯中必要的准备阶段。中国的博士后制度虽然起步晚，但已形成独具特色和相对独立、完善的人才培养和使用机制，成为造就高水平人才的重要途径，它已经并将继续为推进中国的科技教育事业和经济发展发挥越来越重要的作用。

中国博士后制度实施之初，国家就设立了博士后科学基金，专门资助博士后研究人员开展创新探索。与其他基金主要资助“项目”不同，博士后科学基金的资助目标是“人”，也就是通过评价博士后研究人员的创新能力给予基金资助。博士后科学基金针对博士后研究人员处于科研创新“黄金时期”的成长特点，通过竞争申请、独立使用基金，使博士后研究人员树立科研自信心，塑造独立科研人格。经过 30 年的发展，截至 2015 年底，博士后科学基金资助总额约 26.5 亿元人民币，资助博士后研究人员 5 万 3 千余人，约占博士后招收人数的 1/3。截至 2014 年底，在我国具有博士后经历的院士中，博士后科学基金资助获得者占 72.5%。博士后科学基金已成为激发博士后研究人员成才的一颗“金种子”。

在博士后科学基金的资助下，博士后研究人员取得了众多前沿的科研成果。将这些科研成果出版成书，既是对博士后研究人员创新能力的肯定，也可以激发在站博士后研究人员开展创新研究的热情，同时也可以使博士后科研成果在更广范围内传播，更好地为社会所利用，进一步提高博士后科学基金的资助效益。

中国博士后科学基金会从 2013 年起实施博士后优秀学术专著出版资助工作。经专家评审，评选出博士后优秀学术著作，中国博士后科学基金会资助出版费用。专著由科学出版社出版，统一命名为《博士后文库》。

资助出版工作是中国博士后科学基金会“十二五”期间进行基金资助改革的一项重要举措，虽然刚刚起步，但是我们对它寄予厚望。希望通过这项工作，使博士后研究人员的创新成果能够更好地服务于国家创新驱动发展战略，服务于创新型国家的建设，也希望更多的博士后研究人员借助这颗“金种子”迅速成长为国家需要的创新型、复合型、战略型人才。



中国博士后科学基金会理事长

FOREWORD

This book is the outcome of years of collaboration among the three coauthors. The collaboration resulted in a number of papers, whose most significant results compose the major portion of this book. The principal coauthor is Dr. Yiqi Zhang. He joined the postdoctoral research station of Electronic Science and Technology at Xi'an Jiaotong University in 2012, under the supervision of Prof. Yanpeng Zhang, who is in charge of the "Quantum Control of Multi-Wave Mixing-Key Scientific and Technological Innovation Team" of the Shaanxi Province. In the past few decades, Prof. Y.P. Zhang and his research team generated a lot of notable scientific results, owing to a solid foundation and inspiring academic atmosphere, which provided abundant nourishment for quick development of younger researchers. Prof. Milivoj Belić is team's international collaborator, who started collaborating with Dr. Y.Q. Zhang during his stay in Germany. Prof. Belić is professor in physics at the Texas A&M University at Qatar and the team leader of the Qatar Nonlinear Science Initiative. Thanks to the strong support for research by the Qatar National Research Fund, Dr. Y.Q. Zhang was able to visit Doha for extended periods in the past few years.

The book is a summary of coauthors research during the past five years, and the results are obtained and published jointly. The results involve not only analytical analysis but also extensive numerical simulations. The book covers a series of research topics in photonics of high current interest, including photonic topological insulators, optical rogue waves, Airy beams, Talbot effect, optical vortices, and other. The contents of the book are as follows.

In Chapter 1, the theory of physical models that will be expounded in this book is briefly introduced, which includes the derivation of the paraxial wave equation and the development of susceptibilities in atomic vapors.

In Chapter 2, the spatial periodic modulation of light is considered. By using the three-beam interference method and nonlinear phase shift modulation, we first investigate the photonic topological insulators in atomic vapors. Secondly, we investigate the Talbot effect resulting from periodically modulated multi-wave mixing. Thirdly, we discuss the nonlinear Talbot effect of rogue waves, which is a real nonlinear optical effect. The effects mentioned are generated in atomic or bulk dielectric media. In the last section of the Chapter, we discuss spatial light modulation in discrete systems, resulting in the proposal of a beam combiner and splitter.

In Chapter 3, the role of nonlinearities in light modulation is discussed. We first demonstrate that optical vortices (as well as vortex pairs) appear in atomic vapors during propagation, when the third- and fifth-order nonlinearities (the so-called cubic-quintic competing nonlinearities) are considered. Secondly, the interaction of incoherent solitons in a photorefractive medium is investigated, in which the nonlinearity is saturable. The last topic discussed in this Chapter is that of azimuthons, which connect necklace solitons and optical vortices. In this part, we consider a weak Kerr nonlinearity, but with deep potentials of different symmetries.

In Chapter 4, the propagation dynamics of some novel optical beams is investigated, including Airy, Bessel-Gauss, and Laguerre-Gauss beams, as well as Fresnel diffraction patterns. In addition, Mathieu and Weber beams are discussed from the same point of view. The media in which these beams propagate include linear media, Kerr and saturable nonlinear media, and media with harmonic potential. We find that spatial solitons can be formed during interaction of Airy beams in nonlinear media, but the solitons do not exhibit the self-accelerating property. We also show how Airy wave functions, Airy breathers and (dual) Airy-Talbot effect can be considered from a unified viewpoint. Based on the harmonic potential model, we discover a new class of self-Fourier beams – the beams whose Fourier transform are the beams themselves. In addition, if the harmonic potential is inserted into the fractional Schrödinger equation, we show that a Gaussian beam propagates along a zigzag and a funnel-like path in one and two dimensions.

In Chapter 5, a summary of the book is presented, with an outlook on future investigations.

Such an arrangement of the book not only provides for a relative independence of topics discussed in different chapters, but also allows for immanent connections among the topics. We believe this book may become a useful reference for researchers in photonics. Despite our careful exposition, mistakes cannot be avoided in a book addressing very recent research advances. Therefore, comments and criticisms are welcome.

In addition to the support from the China Postdoctoral Science Foundation (Nos. 2014T70923 and 2012M521773), the project was also supported by the National Basic Research Program of China (No. 2012CB921804), the National Natural Science Foundation (Nos. 61308015 and 11474228), the Key Scientific and Technological Innovation Team of Shaanxi Province (No. 2014KCT-10), the Natural Science Foundation of Shaanxi province (No. 2014JQ8341), the Fundamental Research Funds for the Central Universities (No. xjj2013089), and the National Priorities Research Program (projects No. 6-021-1-005 and 09-462-1-074) from the Qatar National Research Fund (a member of the Qatar Foundation).

Last but not least, the coauthors would like to express sincere appreciation to Prof. Song Jianping, Dr. Li Changbiao, Dr. Zheng Huaibin, Dr. Chen Haixia, Dr. Wang Zhiguo, Dr. Wang Ruimin, Dr. Wu Zhenkun, Dr. Petrović Milan S, Dr. Wen Feng, Dr. Zhang Zhaoyang, Mr. Liu Xing, Miss Zhong Hua, Prof. Lu Keqing, Prof. Li Yuanyuan, and other scientists who generously helped us in obtaining research results exposed in this book. We also express our gratitude to Prof. Xiao Min from Arkansas University, Prof. Huang Tingwen from Texas A&M University at Qatar, and Prof. Zhong Weiping from Shunde Polytechnic. Finally, special thanks go to the Postdoctoral Office at Xi'an Jiaotong University, the China Postdoctoral Science Foundation, and the Science Press.

Zhang Yiqi
Xi'an Jiaotong University
Xi'an 710049, China
Email: zhangyiqi@mail.xjtu.edu.cn
Homepage: http://zhangyiqi.gr.xjtu.edu.cn

Belić Milivoj
Texas A&M University at Qatar
23874 Doha, Qatar
Email: milivoj.belic@qatar.tamu.edu
Homepage: http://science.qatar.tamu.edu/FacultyandStaff/Pages/default.aspx

Zhang Yanpeng
Xi'an Jiaotong University
Xi'an 710049, China
Email: ypzhang@mail.xjtu.edu.cn
Homepage: http://ypzhang.gr.xjtu.edu.cn

CONTENTS

Chapter 1

BASIC THEORY

1.1 The paraxial wave equation

In the framework of classical electromagnetic theory, light is an electromagnetic wave, which satisfies Maxwell's equations. In SI units, Maxwell's equations have the form

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{1.1}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{1.2}$$

$$\nabla \cdot \boldsymbol{D} = \rho, \tag{1.3}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{1.4}$$

where $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, and $\boldsymbol{B}$ are the vectors of electric field, electric displacement, magnetic field, and magnetic flux, respectively. In a charge-free dielectric medium, the electric charge density is $\rho = 0$ and the electric current density is $\boldsymbol{J} = 0$. In non-magnetic media, there are also constitutive relations

$$\boldsymbol{B} = \mu_0 \boldsymbol{H}, \tag{1.5}$$

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}, \tag{1.6}$$

where $\boldsymbol{P}$ is the dielectric polarization, ϵ_0 is the permittivity in vacuum, and μ_0 is the permeability in vacuum. Considering that the polarization can be divided into a linear polarization $\boldsymbol{P}_{\mathrm{L}}$ and a nonlinear polarization $\boldsymbol{P}_{\mathrm{NL}}$, and using the relation $\nabla \times \nabla \times \boldsymbol{A} = -\nabla^2 \boldsymbol{A} + \nabla \cdot \boldsymbol{A}$, one obtains the wave equation from Maxwell's equations

$$\nabla^2 \boldsymbol{E} = \mu_0 \frac{\partial^2}{\partial t^2} \left(\epsilon_0 \boldsymbol{E} + \boldsymbol{P}_{\mathrm{L}} + \boldsymbol{P}_{\mathrm{NL}} \right). \tag{1.7}$$

Assuming that the light field is monochromatic and harmonic, i.e.,

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{E}(x, y, z) \exp(-\mathrm{i}\omega t),$$

Eq. (1.7) can be written as

$$\nabla^2 \boldsymbol{E} + \omega^2 \mu_0 \left(\epsilon_0 \boldsymbol{E} + \boldsymbol{P}_{\mathrm{L}} + \boldsymbol{P}_{\mathrm{NL}}\right) = 0. \tag{1.8}$$

According to the spatial symmetry properties, the linear and nonlinear polarizations can be expanded in a series

$$\begin{aligned} \boldsymbol{P}_{\mathrm{L}} &= \epsilon_0 \chi^{(1)} \boldsymbol{E}, \\ \boldsymbol{P}_{\mathrm{NL}} &= \epsilon_0 (\chi^{(3)} |\boldsymbol{E}|^2 + \chi^{(5)} |\boldsymbol{E}|^4 + \cdots) \boldsymbol{E}. \end{aligned}$$

If one defines

$$\begin{aligned} \epsilon &= 1 + \chi \\ &= 1 + \chi^{(1)} + \chi^{(3)} |\boldsymbol{E}|^2 + \chi^{(5)} |\boldsymbol{E}|^4 + \cdots , \end{aligned}$$

then Eq. (1.8) can be rewritten as

$$\nabla^2 \boldsymbol{E} + k_0^2 n^2 \boldsymbol{E} = 0, \tag{1.9}$$

where $\chi^{(1)}$ is the linear susceptibility, $\chi^{(m)}$ is the m^{th} order nonlinear susceptibility, $k_0 = \omega / c$, $c = \sqrt{1/\epsilon_0 \mu_0}$ is the light speed in vacuum, $n = \sqrt{\epsilon}$ is the refractive index of the medium. Eq. (1.9) is known as the Helmholtz equation.

Assuming that the envelope of the light field $\psi(x, y, z)$ varies slowly along the propagation direction $\boldsymbol{z}$, i.e., adopting the slowly-varying envelope approximation [1]

$$\left| \frac{\partial^2 \psi}{\partial z^2} \right| \ll \left| k \frac{\partial \psi}{\partial z} \right|,$$

the light field can be written as

$$E = \psi(x, y, z) \exp(\mathrm{i} k z), \tag{1.10}$$

where $k = n_0 k_0$, and $n_0 = n(x \to \infty,\ y \to \infty)$ is the background refractive index of the medium. Plugging Eq. (1.10) into Eq. (1.9), one obtains

$$\mathrm{i} \frac{\partial \psi}{\partial z} + \frac{1}{2k} \nabla^2 \psi + \frac{k}{n_0} \delta n \psi = 0, \tag{1.11}$$

where $\nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2$ is the two-dimensional transverse Laplacian, and $\delta n = n - n_0$ is the refractive index change. In Eq. (1.11), δn indicates the response of the medium to light, and determines the linear/nonlinear type the medium[2]. In this manner, one obtains the paraxial wave equation – Eq. (1.11), the form of which is the same as the Schrödinger equation in quantum mechanics[3].

If the susceptibility is small in a system, the refractive index change can be approximately written as

$$\delta n \approx \frac{1}{2}\chi. \tag{1.12}$$

Plugging Eq. (1.12) into Eq. (1.11), one obtains the wave equation used in atomic vapors[4−7]

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \frac{1}{2k}\nabla^2\psi + \frac{k}{2n_0}\chi\psi = 0. \tag{1.13}$$

Eq. (1.11) is an equation with real physical quantities. However, for mathematical ease, it is convenient to deal with dimensionless equations. Replacing x, y, and z in Eq. (1.11) with the dimensionless coordinates xr_0, yr_0, and zkr_0^2, one obtains the dimensionless equation

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \frac{1}{2}\nabla^2\psi + \frac{k^2r_0^2}{n_0}\delta n\psi = 0, \tag{1.14a}$$

in which r_0 is the beam width, and kr_0^2 is known as the Rayleigh length. Sometimes, $2zkr_0^2$ is used as the Rayleigh length, so that the dimensionless equation then reads

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \nabla^2\psi + 2\frac{k^2r_0^2}{n_0}\delta n\psi = 0. \tag{1.14b}$$

1.2 Susceptibilities in atomic vapors

Let us consider a close-cycled $(n+1)$-level cascade system [8], as shown in Fig. 1.1. The transition from state $|i-1\rangle$ to state $|i\rangle$ is driven by two laser fields $E_i(\omega, \boldsymbol{k}_1)$ and $E_i'(\omega, \boldsymbol{k}_1')$, with Rabi frequencies G_i and G_i', respectively. The Rabi frequencies are defined as

$$G_i = E_i\mu_{ij}/\hbar \text{ and } G_i' = E_i'\mu_{ij}/\hbar,$$

where μ_{ij} is the transition dipole moment between levels $|i\rangle$ and $|j\rangle$. The fields E_n and E_n' (of the same frequency) propagate along beams 2 and 3, respectively, with a small angle θ between them (Figure 1.1(a)). The fields E_2, E_3 to E_{n-1} propagate along the direction of beam 2, while a weak probe field E_1 (the beam 1) propagates in the opposite direction to beam 2. The simultaneous interactions of the multi-level atoms with fields E_1, E_2 to E_n will induce atomic coherence between states $|0\rangle$ and $|n\rangle$ through the resonant n-photon transitions. This n-photon coherence is then probed by the field E_n' and, as a result, a $2n$-wave mixing signal of frequency ω_1 in beam 4 is generated almost exactly opposite to the direction of beam 3, satisfying the phase-matching condition $\boldsymbol{k}_{2n} = \boldsymbol{k}_1 + \boldsymbol{k}_n - \boldsymbol{k}_n'$.

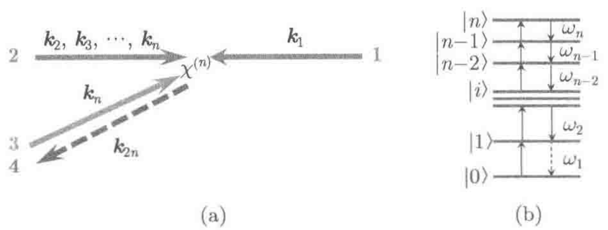


Figure 1.1 (a) Schematic diagram for the phase-conjugate $2n$-wave mixing process; (b) Energy-level diagram for $2n$-wave mixing in a close-cycled $(n+1)$-level cascade system

Using the master equation for the evolution of this system, one can write [1]

$$\frac{\partial \hat{\rho}(t)}{\partial t} = \frac{1}{\mathrm{i}\hbar}[\hat{H}_0 + \hat{H}_1(t), \hat{\rho}(t)] - \Gamma \hat{\rho}$$

where $\hat{H}_1 = -E\hat{\mu}$ is the dipole interaction Hamiltonian. Then, we can expand the density operator $\hat{\rho}(t)$ and write

$$\hat{\rho}(t) = \hat{\rho}^{(0)}(t) + \hat{\rho}^{(1)}(t) + \hat{\rho}^{(2)} + \cdots + \hat{\rho}^{(r)}(t) + \cdots \tag{1.15}$$

By introducing this expansion into the initial master equation, the density-matrix equation takes the form

$$\begin{aligned} &\mathrm{i}\hbar\frac{\partial}{\partial t}[\hat{\rho}^{(0)}(t) + \hat{\rho}^{(1)}(t) + \hat{\rho}^{(2)} + \cdots + \hat{\rho}^{(r)}(t) + \cdots] \\ =&[\hat{H}_0 + \hat{H}_1, \hat{\rho}^{(0)}(t) + \hat{\rho}^{(1)}(t) + \hat{\rho}^{(2)} + \cdots + \hat{\rho}^{(r)}(t) + \cdots] \\ &- \mathrm{i}\Gamma\hbar[\hat{\rho}^{(0)}(t) + \cdots + \hat{\rho}^{(r)}(t) + \cdots]. \end{aligned} \tag{1.16}$$

Separating the density operator in Eq. (1.16) with the same order, one obtains a series of equations

$$\begin{cases} \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho}^{(0)}(t) = [\hat{H}_0, \hat{\rho}^{(0)}(t)] - \mathrm{i}\hbar\Gamma[\hat{\rho}^{(0)}(t)], \\ \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho}^{(1)}(t) = [\hat{H}_0, \hat{\rho}^{(1)}(t)] + [\hat{H}_1, \hat{\rho}^{(0)}(t)] - \mathrm{i}\hbar\Gamma\hat{\rho}^{(1)}(t), \\ \qquad\cdots \\ \mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho}^{(r)}(t) = [\hat{H}_0, \hat{\rho}^{(r)}(t)] + [\hat{H}_1, \hat{\rho}^{(r-1)}(t)] - \mathrm{i}\hbar\Gamma\hat{\rho}^{(r)}(t). \end{cases} \tag{1.17}$$

Then, one can find the series $\rho^{(0)}, \cdots, \rho^{(r)}$ by solving the above equations step by step (from the lower to the higher orders). According to the density-matrix equations, one can write the above dynamic equations in the matrix form, with the matrices given by

$$\hat{\mu}=\begin{bmatrix}0 & \mu_1 & 0 & \cdots & 0 & 0\\ \mu_1 & 0 & \mu_2 & \cdots & 0 & 0\\ 0 & \mu_2 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & \mu_n\\ 0 & 0 & 0 & \cdots & \mu_n & 0\end{bmatrix},\ \hat{H}_0=\begin{bmatrix}E_0 & 0 & \cdots & 0\\ 0 & E_1 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & E_n\end{bmatrix},$$

$$\hat{\rho}^{(r)}=\begin{bmatrix}\rho_{00}^{(r)} & \rho_{01}^{(r)} & \cdots & \rho_{0,n}^{(r)}\\ \rho_{10}^{(r)} & \rho_{11}^{(r)} & \cdots & \rho_{1,n}^{(r)}\\ \vdots & \vdots & & \vdots\\ \rho_{n,0}^{(r)} & \rho_{n,1}^{(r)} & \cdots & \rho_{n,n}^{(r)}\end{bmatrix},\ \Gamma\hat{\rho}^{(r)}=\begin{bmatrix}\Gamma_0\rho_{00}^{(r)} & \Gamma_{10}\rho_{01}^{(r)} & \cdots & \Gamma_{n,0}\rho_{0,n}^{(r)}\\ \Gamma_{10}\rho_{10}^{(r)} & \Gamma_1\rho_{11}^{(r)} & \cdots & \Gamma_{n,1}\rho_{1,n}^{(r)}\\ \vdots & \vdots & & \vdots\\ \Gamma_{n,0}\rho_{n,0}^{(r)} & \Gamma_{n,1}\rho_{n,1}^{(r)} & \cdots & \Gamma_n\rho_{n,n}^{(r)}\end{bmatrix}.$$

For the diagonal element ρ_{ii}, Γ_i represents the longitudinal relaxation rate. However, for the off-diagonal element ρ_{ij}, Γ_{ij} is the transverse relaxation rate. μ_i is the transition dipole moment. Then, the dynamic equation can be written as

$$\begin{aligned}&[\hat{H}_0,\hat{\rho}^{(r)}]\\ =&\hat{H}_0\hat{\rho}^{(r)}-\hat{\rho}^{(r)}\hat{H}_0\\ =&\begin{bmatrix}0 & \rho_{01}^{(r)}(E_0-E_1) & \cdots & \rho_{0,n}^{(r)}(E_0-E_n)\\ \rho_{10}^{(r)}(E_1-E_0) & 0 & \cdots & \rho_{1,n}^{(r)}(E_1-E_n)\\ \vdots & \vdots & & \vdots\\ \rho_{n-1,0}^{(r)}(E_{n-1}-E_0) & \rho_{n,1}^{(r)}(E_n-E_1) & \cdots & 0\end{bmatrix},\\ &[\hat{H}_1,\hat{\rho}^{(r-1)}]\\ =&-E[\hat{\mu}_1,\hat{\rho}^{(r-1)}]=-E[\hat{\mu}_1\hat{\rho}^{(r-1)}-\hat{\rho}^{(r-1)}\hat{\mu}_1]\\ =&-E\begin{bmatrix}\left(-\rho_{01}^{(r-1)}+\rho_{10}^{(r-1)}\right)\mu_1 & \cdots & -\rho_{0,n-2}^{(r-1)}\mu_{n-1}+\rho_{1,n-1}^{(r-1)}\mu_1\\ \left(-\rho_{11}^{(r-1)}+\rho_{00}^{(r-1)}\right)\mu_1+\rho_{20}^{(r-1)}\mu_2 & \cdots & -\rho_{1,n-2}^{(r-1)}\mu_{n-1}+\rho_{0,n-1}^{(r-1)}\mu_1+\rho_{2,n-1}^{(r-1)}\mu_2\\ \vdots & & \vdots\\ -\rho_{n,1}^{(r-1)}\mu_1+\rho_{n-1,0}^{(r-1)}\mu_n & \cdots & \left(-\rho_{n,n-1}^{(r-1)}+\rho_{n-2,n}^{(r-1)}\right)\mu_n\end{bmatrix}.\end{aligned}$$

According to the equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho}_{ij}^{(r)}(t)=[\hat{H}_0,\hat{\rho}_{ij}^{(r)}(t)]+[\hat{H}_1,\hat{\rho}_{ij}^{(r-1)}(t)]-\mathrm{i}\hbar\Gamma_{ij}\hat{\rho}_{ij}^{(r)}(t),$$

one obtains

$$\begin{cases} \dfrac{\partial\rho_{10}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{10}(E_1 - E_0) - E(-\mu_1\rho_{11} + \mu_1\rho_{00} + \mu_2\rho_{20})] - \Gamma_{10}\rho_{10}, \\ \dfrac{\partial\rho_{20}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{20}(E_2 - E_0) - E(-\mu_1\rho_{21} + \mu_2\rho_{10} + \mu_3\rho_{30})] - \Gamma_{20}\rho_{20}, \\ \qquad\cdots \\ \dfrac{\partial\rho_{n-1,0}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{n-1,0}(E_{n-1} - E_0) - E(-\mu_1\rho_{n-1,1} + \mu_{n-1}\rho_{n-2,0} + \mu_n\rho_{n,0})] \\ \qquad - \Gamma_{n-1,0}\rho_{n-1,0}, \\ \dfrac{\partial\rho_{n,0}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{n,0}(E_n - E_0) - E(-\mu_1\rho_{n,1} + \mu_n\rho_{n-1,0})] - \Gamma_{n,0}\rho_{n,0}. \end{cases}$$

As an example of this general description, we now consider a five-level folded atomic system, as shown in Fig. 1.2.

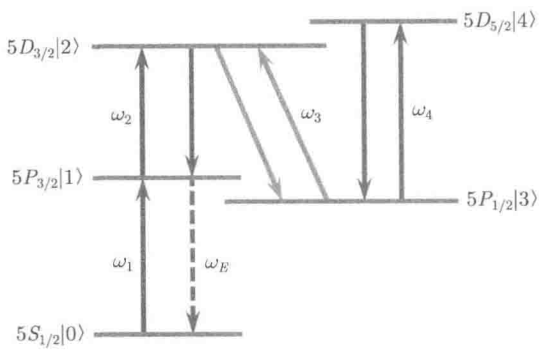


Figure 1.2 Energy-level diagram of a close-cycled (folded) five-level atomic system

Based on the above derivation, the density-matrix equations are

$$\begin{cases} \dfrac{\partial\rho_{10}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{10}(E_1 - E_0) - E(\mu_1\rho_{00} + \mu_2\rho_{20} - \mu_1\rho_{11})] - \Gamma_{10}\rho_{10}, \\ \dfrac{\partial\rho_{20}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{20}(E_2 - E_0) - E(\mu_2\rho_{10} + \mu_3\rho_{30} - \mu_1\rho_{21})] - \Gamma_{20}\rho_{20}, \\ \dfrac{\partial\rho_{30}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{30}(E_3 - E_0) - E(\mu_3\rho_{20} + \mu_4\rho_{40} - \mu_1\rho_{31})] - \Gamma_{30}\rho_{30}, \\ \dfrac{\partial\rho_{40}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\rho_{40}(E_4 - E_0) - E(\mu_4\rho_{30} - \mu_1\rho_{41})] - \Gamma_{40}\rho_{40}. \end{cases} \tag{1.18}$$

In the bare-state picture, the equations of motion for the atomic polarizations and populations (atomic responses) are considered up to different orders of Liouville pathways that provide a diagrammatic representation to designate the time evolution of the density matrix of the system [9]. Thus, we can employ perturbation theory to calculate the density-matrix elements. In this five-level system, the perturbation chains (i.e. Liouville pathways within the perturbation theory) are written