Computational Mechanics Series

Multiphysics Modeling:
Numerical Methods and Engineering Applications

多物理场仿真：
数值方法及工程应用（英文版）

张群　岑松　著

Qun Zhang　Song Cen

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内 容 简 介

本书为英文版。书中详细介绍了工程中常见的结构、流体、温度、电磁、噪声等物理场间的耦合问题，论述了强耦合方法、弱耦合方法以及跨程序之间的耦合方法。对不同耦合边界条件及其数值实现方法进行了深入讨论，涉及流-固耦合、流-固-温度场耦合、电磁-流-固-热耦合等复杂耦合问题。书中还列举了汽车工程、航空航天工程、旋转机械、微机电系统、生物医学工程等领域的20多个实际耦合分析案例。本书提供了从耦合理论、方法、软件开发到工程实践的较为全面综合的知识和工程实践体系，可供相关学科和行业的科研人员、工程技术人员、软件开发人员阅读参考，也可作为理工科相关课程的研究生教材。

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Preface

The readers of this book are the researchers and engineers who are interested in the numerical methodology study, code implementation and engineering applications in multidisciplinary problems. This book can also be used as a textbook for graduate students and high level undergraduate students in mechanical engineering, automotive engineering, aerospace engineering, civil engineering, biomechanical engineering, and many other areas.

Research and engineering applications regarding multiphysics simulations have been given great attention over the last decade. This book describes the basic principles and methods for multiphysics modeling, covering related areas of physics, such as structure mechanics, fluid dynamics, heat transfer, electromagnetic, and acoustics fields. Although the fundamental equations for each and every physics model are presented in this book, the main focus will be on the coupling related terms and conditions, as well as the nonlinearity and stabilization issues.

In this book, the coupling problems are classified into different categories, namely: (1) essential coupling; (2) production term coupling; (3) natural boundary condition coupling; (4) constitutive equation coupling; and (5) analysis domain coupling by physical characteristics, which include strong coupling problems and weak coupling problems, by the level of the coupling. All of the possible interface coupling conditions and load transfer conditions among these five physics models are listed, and the resulted coupled equations are presented. The Direct Matrix Assembly (DMA) method, Direct Interface Coupling (DIC) method, Multipoint Constraint (MPC) equation-based, Lagrangian Multiplier (LM) based and Penalty Method (PM) based strong coupling methods are proposed for different types of strongly coupled problems. Background theories, algorithms, key technologies, and code implementation for inter-solver weak coupling methods are also covered and discussed in details.

The challenges and important topics in multiphysics simulation are also presented: the nonlinearities and numerical stabilization in spatial and time domain; multiphysics simulation of rotating machinery; moving boundary problems for nonstructural physics models; parallel computing for large scale multiphysics simulation, etc.

This book systematically discusses about the multiphysics modeling among fluid, structure, thermal, electromagnetic, and acoustics problems. The fundamental equations, numerical schemes as well as the strategies and procedures for code implementation are presented. Most of the technologies presented in this book have been implemented in general purposed multiphysics simulation software INTESIM. More than 20 valuable engineering applications in automotive, aerospace, MEMS device, rotating machinery, and biomedical engineering etc. are presented in this book.

Organization of chapters

In Chapter 1, we briefly review the fundamental equation for fluid dynamics, structure mechanic, thermal, electromagnetic analysis, and acoustics. Special emphasis is put on the coupling terms in each equation, as well as the discretization and stabilization algorithm.

In Chapter 2, we clarify the coupling types and coupling characteristics among different physics models, and also review and discuss about appropriate coupling methods for different coupling problems.

In Chapter 3, we discuss about the coupling methods, which include the strong coupling method, general weak coupling method and intersolver-based weak coupling method.

The morphing and automatic re-meshing scheme is presented in Chapter 4 for the nonstructural physics with moving boundary analysis and coupled physics simulation with deformed structure.

The stabilization method in space and time domain to cope with the coupling nonlinearity and convergence issues in each physics model, are discussed in Chapter 5.

The multiphysics coupling problems of rotating machinery are addressed in Chapter 6.

In Chapter 7, the parallel algorithm for strong and weak coupling methods is introduced.

Three general fluid–structure interaction analyses will be presented in Chapter 8.

Multiphysics simulations in automotive engineering, aerospace engineering, MEMS devices, turbine machinery and biomechanical engineering are presented in Chapter 9, Chapter 10, Chapter 11, Chapter 12, and Chapter 13, respectively.

In Chapter 14, the FSI simulation of a sensor device used in civil engineering, and an acoustic-structure coupling problem of a closed cube metal box are presented.

In Chapter 15, an overview of the commercial multiphysics software is given and the features of code implementation for multiphysics coupling is presented.

This book covers five different physics fields, namely: fluid, structure, thermal, electromagnetic, and acoustics fields. We tried our best to use common and consistent styles of statement and symbols throughout this book to make it easy to read.

This is the first edition of the book. Due to the wide coverage of the book and limitation of our knowledge, if you find any mistakes and errors in this edition, please kindly give us the feedback. We will appreciate your tolerance and efforts and correct them in the next edition.

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Many of our collaborators: Prof Liu Bin and Dr Zhu Baoshan, from Tsinghua University, Prof Guan Zhenqun and Prof Liu Jun from Dalian University of Technology, Prof Li Hongguang from Shanghai Jiaotong University, and Prof Hu Qiya from the Chinese Academy of Science, for their advice and the contributions to the development of INTESIM technology.

We would like to say thanks to all our colleagues from INTESIM (Dalian) Co. Ltd. who have worked hard to prepare this book. Computational technology development and engineering examples: Dr Jiang Peng. Engineering examples: Mr Han Yepeng, Ms Huang Xiaoxiao, Mr Sun Xinghua, and Ms Xu Yu.

Many thanks to our colleagues and students who helped us in checking and verifying the English and prepare graphs, tables, etc: Ms Bai Liang, Dr Li Jianqiao, Mr Wu Xiaoming, Mr Xu Shuoyuan, Mr Zhou Xiaogang, and Mr Zhu Rubin.

We would not have the chapter on multiphysics simulation in aerospace engineering without the help of our collaborator, Prof Liu Jun from Dalian University of Technology.

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Contents

The physics models

1

Chapter Outline

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1.1 Heat flow fundamentals

1.1.1 Basic equations

The solution for heat flow analysis is to find the temperature distribution $T(X,t) \in \mathcal{S}_T$, so that the following governing Equation (1.1) is satisfied.

$$\frac{\partial(\rho c_v T)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho c_v T v_i - k\frac{\partial T}{\partial x_i}\right) = \overleftarrow{q^B}, \text{ in } \Omega_t^T \tag{1.1}$$

The primary variable is temperature T in the infinite-dimensional space of $\mathcal{S}_T$. The first part on the left-hand side of Equation (1.1) is the time derivative term, the second term is the convection term caused by fluid flow, and the third term is the diffusion term. The source term of heat generation is on the right-hand side of Equation (1.1).

The material properties needed to be decided are; mass density, ρ; specific heat, c_v; and thermal conductivity, k. For fully incompressible flow, ρ is assumed to be constant, but for slightly compressible and low-speed compressible flow, we assume $\rho = \rho(p)$ and $\rho = \rho(p, T)$, respectively. Also, we assume specific heat c_v and conductivity k to be either constant or a function of temperature. $\boldsymbol{v}$ is the convective velocity, and q^B is the heat generation from other physics models in multiphysics simulation.

Remark 1.1: q^B is the source term of heat generation that may come from thermo-elastic damping, fluid viscous heat, or electro or electromagnetic heat in multiphysics simulation. On the other hand, temperature T as output may affect the material properties or other quantities of the corresponding physics models.

Symbol $\overleftarrow{\ }$ indicates the value that is received from other physics model, and $\overrightarrow{\ }$ represents the value transferred to other physics model in multiphysics simulation. Symbol $\overleftrightarrow{\ }$ means the value can be either sent out to or received from other physics model in multiphysics simulation.

1.1.2 Boundary conditions

Three types of boundary conditions are considered:

1. Specified temperature on Γ_g:

$$T = \overleftrightarrow{T^s} \quad \text{on} \quad \Gamma_g \tag{1.2}$$

Here, T^s is the specified temperature.

2. Specified heat flow on Γ_{q1}:

$$k \cdot \frac{(n_i \partial T)}{(\partial x_i)} = \overleftrightarrow{q^s} \quad \text{on} \quad \Gamma_{q1} \tag{1.3}$$

where, q^s is the specified heat flow and n_i is the component of the unit normal direction $\boldsymbol{n}$.

Remark 1.2: The boundary conditions: Equations (1.2) and (1.3) can be used in conjugate heat transfer coupling.

3. Specified convection surfaces acting over surface Γ_{q2}:

$$k\cdot\frac{(n_i\ \partial T)}{(\partial x_i)} = h_f(T_B - T_S) \quad \text{on} \quad \Gamma_{q2} \tag{1.4}$$

Where; h_f, is the convective heat transfer coefficient; T_B, is the bulk temperature of the adjacent physics model; and T_S, is the temperature at the surface of the model.

Remark 1.3: The positive specified heat flow is into the boundary.

1.1.3 Weak forms of the thermal equation

The weak form of heat flow equations is given in Equation (1.5) as follows:
find temperature $T \in \mathcal{S}_T$, such that $\forall w_T \in \mathcal{W}_T$:

$$\int_\Omega w_T\left[\frac{\partial(\rho c_v T)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho c_v T v_i - k\frac{\partial T}{\partial x_i}\right)\right]\mathrm{d}\Omega = \int_\Omega w_T q^B \mathrm{d}\Omega. \tag{1.5}$$

Here, w_T is the test function of temperature T in infinite-dimensional space $\mathcal{W}_T$.
The semi-discrete Galerkin finite element formulation of the heat flow problem is stated as:
find $T^h \in \mathcal{S}_T^h$, such that $\forall w_T^h \in \mathcal{W}_T^h$:

$$\int_\Omega w_T^h\left[\frac{\partial(\rho c_v T^h)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho c_v T^h v_i^h - k\frac{\partial T^h}{\partial x_i}\right)\right]\mathrm{d}\Omega = \int_\Omega w_T^h q^{B^h} \mathrm{d}\Omega \tag{1.6}$$

Where, $\mathcal{S}_T^h$ and $\mathcal{W}_T^h$ are the sets of finite-dimensional trial and test functions for temperature, respectively.

1.1.4 The shape functions for FEM

The basic shape function for the coordinate:

$$\boldsymbol{x}(\xi) = \sum_{i=1}^{n_{en}} N_i^E(\xi)\boldsymbol{x}_i = N_i^E(\xi)\boldsymbol{x}_i. \tag{1.7}$$

Here; $N_i^E(\xi)$, is the shape function at node i; ξ, is the element local coordinate; and $\boldsymbol{x}_i$, is the nodal coordinate at node i; n_{en}, is the number of nodes per element, respectively.
The interpolation for temperature variables:

$$T(\xi) = \sum_{i=1}^{n_{en}} N_i^{\mathrm{T}} T_i = N_i^{\mathrm{T}} T_i. \tag{1.8}$$

Here, N_i^T is the shape function for temperature at node i, which is consistent with N_i^E for the class of isoparametrical element. The standard isoparametrical element can be used for the temperature interpolation, for example, the low order 4-node tetrahedral element and 8-node hexahedral element, are the most commonly used for thermal analysis, although the high order 10-node tetrahedral element and 20-node hexahedral element are also useful in achieving better accuracy with the same number of nodes. The other shapes, for example, wedge and pyramid can be used for the transition region between tetrahedral and hexahedral elements. The details about these shape functions can be found in Bathe (2006) and Wang (2000).

1.1.5 Formulations in matrix form

Integrating of Equation (1.6) by part for the third term on the left-hand side, applying boundary conditions of Equations (1.2)–(1.4), and substituting the shape functions into it, one can obtain assembled global Equation (1.9) in matrix form:

$$\boldsymbol{C}\dot{\boldsymbol{T}} + \left(\boldsymbol{K}(\boldsymbol{v})^{tm} + \boldsymbol{K}^{tb} + \boldsymbol{K}^{tc}\right)\boldsymbol{T} = \boldsymbol{Q}^{\text{flux}} + \boldsymbol{Q}^{\text{conv}} + \overline{\boldsymbol{Q}^{g}\left(q^{B}\right)}. \tag{1.9}$$

Where,

$\boldsymbol{C} = \sum_{e=1}^{ne} \boldsymbol{C}_e,\ C_{e(i,j)} = \rho \int_{\Omega_e} c_v N_i^T N_j^T \,\mathrm{d}\Omega$ is the element specific heat matrix.

$\boldsymbol{K}^{tm} = \sum_{e=1}^{ne} \boldsymbol{K}_e^{tm},\ K_{e(i,j)}^{tm} = \rho \int_{\Omega_e} c_v N_i^T v_k N_{j,k}^T \,\mathrm{d}\Omega$ is the element mass transport conductivity matrix.

$\boldsymbol{K}^{tb} = \sum_{e=1}^{ne} \boldsymbol{K}_e^{tb},\ K_{e(i,j)}^{tb} = \int_{\Omega_e} N_{i,m}^T k_{mm} N_{j,m}^T \,\mathrm{d}\Omega$ is the element diffusion conductivity matrix.

$\boldsymbol{K}^{tc} = \sum_{e=1}^{ne} \boldsymbol{K}_e^{tc},\ K_{e(i,j)}^{tc} = \int_{\Gamma_{q2}} h_f N_i^T N_j^T \,\mathrm{d}\Gamma$ is element convection surface conductivity matrix.

$\boldsymbol{Q}^{\text{flux}} = \sum_{f} \boldsymbol{Q}_f^{\text{flux}},\ Q_{f(i)}^{\text{flux}} = \int_{\Gamma_{q1}} N_i^T q^s \,\mathrm{d}\Gamma$ is element surface heat flux vector.

$\boldsymbol{Q}^{\text{conv}} = \sum_{f} \boldsymbol{Q}_f^{\text{conv}},\ Q_{f(i)}^{\text{conv}} = \int_{\Gamma_{q2}} T_B h_f N_i^T \,\mathrm{d}\Gamma$ is the element convection surface heat flow vector.

$\boldsymbol{Q}^{g} = \sum_{e} \boldsymbol{Q}_e^{g},\ Q_{e(i)}^{g} = \int_{\Omega_e} q^B N_i^T \,\mathrm{d}\Omega$ is the element heat generation vector.

Here, $\sum_{e=1}^{ne}$ represents the assembly operator for element matrix or vector with number of *ne* elements.

$\sum_{f}$ represents the assembly operator for element surface loads.

1.1.6 The nonlinearity in thermal analysis

1.1.6.1 Material properties

The material properties depend on the temperature, that is, $k = k(T),\ \ c_v = c_v(T)$.

1.1.6.2 Convection term from computational fluid dynamics (CFD) coupling

The definition of Peclet number for convective heat transfer problem is

$$\text{Pe} = \frac{c_v \rho}{k} \|v\|\, h^e. \tag{1.10}$$

Where, h^e stands for element size.

For a heat flow problem with Pe ≥ 2.0, to avoid oscillation of the solution of the temperature field and to get a better quality matrix equation, the spatial stabilization method may be needed. The streamline upwind Petrov/Galerkin (SUPG) method is presented in the further section.

1.1.7 Stabilization method for convection-dominant transport equations

The SUPG method is used for the convection-dominated heat transfer problem for the case, where Peclet number is higher than 2.0. For details about implementation of the stabilization method for transport equations, please refer Section 1.2.9.3.

1.1.8 Penalty-based thermal contact

The thermal contact problem can be used for thin air gap or other thin-layered problems with different thermal conductivities.

Assuming the thickness of a thin air gap is d, and T_1 and T_2 are the temperatures on the side A and side B of the contact wall, respectively, then the heat flux from T_2 side into T_1 side is:

$$q = k\frac{T_2 - T_1}{d}. \tag{1.11}$$

Here, k is the thermal conductivity, and the heat flux from T_1 side into T_2 side is the negative of q in Equation (1.11). Here we assume the air layer is thin enough, so linear distribution assumption of temperature across the air gap is acceptable.

Then thermal conductance for this gap is:

$$h_f = \frac{k}{d} \tag{1.12}$$

where h_f, as the contact thermal conductance, can be setup as input property.

1.1.8.1 The matrix equation for thermal contact

The heat flow for side A

$$\int_{\Gamma_A} w_A \; h_f \left(T_B - T_A\right) \mathrm{d}\Gamma \tag{1.13}$$

The previous equation can be added into the left-hand side of Equation (1.5) to consider the thermal contact affect from the air gap, and the matrix form can be expressed as:

$$k_{AiAj} = \int_{\Gamma_A} N_{Ai} h_f N_{Aj} \mathrm{d}\Gamma \tag{1.14}$$

$$k_{AiBj} = -\int_{\Gamma_A} N_{Ai} \; h_f N_{Bj} \mathrm{d}\Gamma \tag{1.15}$$

where, Ai is the surface node i on side A and Bj is the surface node j on side B, respectively. $k_{AiAj} T_A + k_{AiBj} T_B$ needs to be added into the left side of Equation (1.9) for side A, and its negative value will be added in the same way for side B.

1.2 Fluid dynamics

1.2.1 Basic equations for fluid flow

We assume the viscous fluid to be isothermal and barotropic (i.e., $F(p,\rho)=0$) and that $\partial p / \partial \rho = B/\rho$, in which B, p, and ρ are fluid bulk modulus, pressure, and fluid density, respectively. The Arbitrary Lagrangian Eulerian (ALE) formulation is usually used to handle moving boundary problems of fluid flow in coupling analysis. The fundamental equations of fluid flow (Zhang and Hisada, 2001) are expressed as:

$$\frac{1}{B} \left.\frac{\partial \vec{p}}{\partial t}\right|_{\chi} + \frac{1}{B} c_i \frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega_t^f \tag{1.16}$$

$$\rho \left.\frac{\partial \vec{v_i}}{\partial t}\right|_{\chi} + \rho c_j \frac{\partial v_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i + \vec{f_i} \quad \text{in} \quad \Omega_t^f \tag{1.17}$$

$$\begin{aligned} &\left.\frac{\partial \left(\rho c_v \vec{T}\right)}{\partial t}\right|_{\chi} + \frac{\partial}{\partial x_i}\left(\rho c_v T v_i - k \frac{\partial T}{\partial x_i}\right) = \\ &\quad 2\mu D^2 + \frac{\partial v_i}{\partial x_i}\left(-p + \lambda \frac{\partial v_i}{\partial x_i}\right) + \vec{q^B} \quad \text{in} \quad \Omega_t^f. \end{aligned} \tag{1.18}$$

The primary variables are pressure p, velocity vector $\boldsymbol{v}$, and temperature T. Here, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{g}$ is the acceleration of gravity, and $\boldsymbol{c} = \boldsymbol{v} - \boldsymbol{v}_m$ is the relative velocity of fluid particles to the ALE coordinates with $\boldsymbol{v}_m$ the mesh velocity. Ω_t^f denotes the spatial thermal fluid domain bounded by the boundary Γ_t^f of interest at any instant t. Here the superscript f stands for the fluid component. λ is the second viscosity, and k is the thermal conductivity. And in Equation (1.18), the strain rate related energy term is expressed as:

$$D^2 = e_{ij}e_{ij} \quad \text{with} \quad e_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}).$$

Remark 1.4: The source term f_i in Equation (1.17) may come from the coupling physics, for example, the electromagnetic force. The outgoing variables are the pressure p and velocity vector $\boldsymbol{v}$ in the coupled physics simulation.

1.2.2 Boundary and initial conditions for fluid flow

The boundary is composed of Γ_t^g and Γ_t^h corresponding to Dirichlet- and Neumann-type boundary conditions, respectively.

$$v_i = \overleftarrow{g_i} \quad \text{on} \quad \Gamma_g^f \tag{1.19}$$

$$\sigma_{ji}n_j = \overline{h_i} \quad \text{on} \quad \Gamma_h^f \tag{1.20}$$

$$T = T^s \quad \text{on} \quad \Gamma_g^T \tag{1.21}$$

$$k \cdot n_i \partial T_i / \partial x_i = q^s \quad \text{on} \quad \Gamma_q^T \tag{1.22}$$

And subject to the following initial conditions:

$$v_i(0) = {}^0v_i \quad \text{on} \quad \Omega^f \tag{1.23}$$

$$p(0) = {}^0p \quad \text{on} \quad \Omega^f \tag{1.24}$$

$$T(0) = {}^0T \quad \text{on} \quad \Omega^f \tag{1.25}$$

The boundary conditions for the energy equation are the same as those in previous sections.

Remark 1.5: Equations (1.19) and (1.20) represent the coupling boundaries in the fluid–structure interaction (FSI) problem with g_i received from the structure and h_i sent to the structure.

1.2.3 The constitutive equation for fluid flow

The fluid is assumed to be Newtonian, and the constitutive equation is:

$$\sigma_{ij} = -p\,\delta_{ij} + \frac{1}{2}\mu\left(v_{i,j} + v_{j,i}\right) \tag{1.26}$$

where δ_{ij} is the component of identity tensor and μ is the dynamic viscosity of the fluid.

The properties for thermal are provided in the previous section.

1.2.4 The weak forms

1.2.4.1 Galerkin formulation for N–S equations

The weak forms for the fluid equations given in Equations (1.16)–(1.18) by the Galerkin method are written as: find pressure $p \in \mathcal{S}_p$, velocity $\boldsymbol{v} \in \mathcal{S}_v$, and temperature $T \in \mathcal{S}_T$, such that $\forall w_p \in \mathcal{W}_p, \forall \boldsymbol{w}_v \in \mathcal{W}_v$, and $\forall w_T \in \mathcal{W}_T$:

$$\int_\Omega w_p \left(\frac{1}{B}\frac{\partial p}{\partial t}\bigg|_\chi + \frac{1}{B} c_i \frac{\partial p}{\partial x_i} + \nabla \cdot \boldsymbol{v} \right) \mathrm{d}\Omega = 0 \tag{1.27}$$

$$\int_\Omega \boldsymbol{w}_v \cdot \rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{c} \cdot \nabla \boldsymbol{v} - \boldsymbol{f} \right) \mathrm{d}\Omega + \int_\Omega \varepsilon(\boldsymbol{w}) : \sigma(\boldsymbol{v}, p)\, \mathrm{d}\Omega = \int_{\Gamma_h} \boldsymbol{w}_v \cdot \boldsymbol{h}\, \mathrm{d}\Gamma \tag{1.28}$$

$$\begin{aligned} &\int_\Omega w_T \left[\frac{\partial(\rho c_v T)}{\partial t} + \nabla \cdot \left(\rho c_v T \boldsymbol{v}_i - k \nabla T \right) \right] \mathrm{d}\Omega = \\ &\quad \int_\Omega w_T \left(2\mu D^2 + q^B + \nabla \cdot \boldsymbol{v}\left(-p + \lambda(\nabla \cdot \boldsymbol{v}) \right) \right) \mathrm{d}\Omega \end{aligned} \tag{1.29}$$

Where $\mathcal{S}_p$, $\mathcal{S}_v$, and $\mathcal{S}_T$ denote the sets of infinite-dimensional trial functions for pressure, velocity, and temperature, respectively. $\mathcal{W}_p$, $\mathcal{W}_v$, and $\mathcal{W}_T$ are the sets of test functions (weighing functions) for the continuity, equilibrium, and energy equations, respectively.

The Galerkin finite element formulation of the fluid equations is stated as follows: find pressure $p^h \in \mathcal{S}_p^h$, velocity $\boldsymbol{v}^h \in \mathcal{S}_v^h$, and temperature $T^h \in \mathcal{S}_T^h$ such that $\forall w_p^h \in \mathcal{W}_p^h, \forall \boldsymbol{w}_v^h \in \mathcal{W}_v^h, \forall w_T^h \in \mathcal{W}_T^h$:

$$\int_\Omega w_p^h \left(\frac{1}{B}\frac{\partial p^h}{\partial t}\bigg|_\chi + \frac{1}{B} c_i^h \frac{\partial p^h}{\partial x_i} + \nabla \cdot \boldsymbol{v}^h \right) \mathrm{d}\Omega = 0 \tag{1.30}$$

$$\int_\Omega \boldsymbol{w}_v^h \cdot \rho \left(\frac{\partial \boldsymbol{v}^h}{\partial t} + \boldsymbol{c}^h \cdot \nabla \boldsymbol{v}^h - \boldsymbol{f}^h \right) \mathrm{d}\Omega + \int_\Omega \varepsilon(\boldsymbol{w}_v^h) : \sigma(\boldsymbol{v}^h, p^h)\, \mathrm{d}\Omega = \int_{\Gamma_h} \boldsymbol{w}_v^h \cdot \boldsymbol{h}^h\, \mathrm{d}\Gamma \tag{1.31}$$