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J.O.Ramsay B.W.Silverman

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Second Edition

函数型数据分析

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《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来,需要数学家淡泊名利并付出更艰苦地努力。另一方面,我们也要从客观上为数学家创造更有利的发展数学事业的外部环境,这主要是加强对数学事业的支持与投资力度,使数学家有较好的工作与生活条件,其中也包括改善与加强数学的出版工作。

从出版方面来讲,除了较好较快地出版我们自己的成果外,引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说,施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书,使我国广大数学家能以较低的价格购买,特别是在边远地区工作的数学家能普遍见到这些书,无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权,一次影印了23本施普林格出版社出版的数学书,就是一件好事,也是值得继续做下去的事情。大体上分一下,这23本书中,包括基础数学书5本,应用数学书6本与计算数学书12本,其中有些书也具有交叉性质。这些书都是很新的,2000年以后出版的占绝大部分,共计16本,其余的也是1990年以后出版的。这些书可以使读者较快地了解数学某方面的前沿,例如基础数学中的数论、代数与拓扑三本,都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点,基础数学类的书以“经典”为主,应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家,例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士,曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然,23本书只能涵盖数学的一部分,所以,这项工作还应该继续做下去。更进一步,有些读者面较广的好书还应该翻译成中文出版,使之有更大的读者群。

总之,我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持,并盼望这一工作取得更大的成绩。

王 元

2005年12月3日

Preface to the Second Edition

This book continues in the footsteps of the First Edition in being a snapshot of a highly social, and therefore decidedly unpredictable, process. The combined personal view of functional data analysis that it presents has emerged over a number of years of research and contact, and has been greatly nourished by delightful collaborations with many friends. We hope that readers will enjoy the book as much as we have enjoyed writing it, whether they are our colleagues as researchers or applied data analysts reading the book as a research monograph, or students using it as a course text.

As in the First Edition, live data are used throughout for both motivation and illustration, showing how functional approaches allow us to see new things, especially by exploiting the smoothness of the processes generating the data. The data sets exemplify the wide scope of functional data analysis; they are drawn from growth analysis, meteorology, biomechanics, equine science, economics and medicine.

“Back to the data” was the heading to the last section of the First Edition. We did not know then how well those words would predict the next eight years. Since then we have seen functional data applications in more scientific and industrial settings than we could have imagined, and so we wanted an opportunity to make this new field accessible to a wider readership than the first volume seemed to permit. Our book of case studies, Ramsay and Silverman (2002), was our first response, but we have known for some time that a new edition of our original volume was also required.

We have added a considerable amount of new material, and considered carefully how the original material should be presented. One main objec-

tive has been, especially when introducing the various concepts, to provide more intuitive discussion and to postpone needless mathematical terminology where possible. In addition we wanted to offer more practical advice on the processing of functional data. To this end, we have added a more extended account of spline basis functions, provided new material on data smoothing, and extended the range of ways in which data can be used to estimate functions. In response to many requests, we have added some proposals for estimating confidence regions, highlighting local features, and even testing hypotheses. Nevertheless, the emphasis in the revision remains more exploratory and confirmatory.

Our treatment of the functional linear model in the First Edition was only preliminary, and since then a great deal of work has been done on this topic by many investigators. A complete overhaul of this material was called for, and the chapters on linear modelling have been completely reworked. On the other hand, our coverage of principal components analysis and canonical correlation still seems appropriate, and not much has been changed. Readers reacted to the later chapters on differential equations as being difficult, and so we have tried to make them a friendlier place to be.

In some places we have opted for an ‘intuitive’ rather than ‘rigorous’ approach. This is not merely because we want our book to be widely accessible; in our view the theoretical underpinnings of functional data analysis still require rather more study before a treatment can be written that will please theoreticians. We hope that the next decade will see some exciting progress in this regard.

We both believe that a good monograph is a personal view rather than a dry encyclopedia. The average of two personal views is inevitably going to be less ‘personal’ than either of the two individual views, just as the average of a set of functions may omit detail present in the original functions. To counteract this tendency, we have ensured that everything we say in our informal and intuitive discussion of certain issues is the view of at least one of us, but we have not always pressed for unanimous agreement!

We owe so much to those who helped us to go here. We would like to repeat our thanks to those who helped with the First Edition: Michal Abrahamowicz, Philippe Besse, Darrell Bock, Catherine Dalzell, Shelly Feran, Randy Flanagan, Rowena Fowler, Theo Gasser, Mary Gauthier, Vince Gracco, Nancy Heckman, Anouk Hoedeman, Steve Hunka, Iain Johnstone, Alois Kneip, Wojtek Krzanowski, Xiaochun Li, Kevin Munhall, Guy Nason, Richard Olshen, David Ostry, Tim Ramsay, John Rice and Xiaohui Wang. We also continue our grateful acknowledgement of financial support from the Natural Science and Engineering Research Council of Canada, the National Science Foundation and the National Institute of Health of the USA, and the British Engineering and Physical Sciences Research Council. The seed for the First Edition, and therefore for the Revised Edition as well, was planted at a discussion meeting of the Royal Statistical Society

Research Section, where one of us read a paper and the other proposed the vote of thanks, not always an occasion that leads to a meeting of minds!

Turning to the Second Edition, Sofia Mosesova and Yoshio Takane read the entire manuscript with an eye to the technical correctness as well as the readability of what they saw, and caught us on many points. David Campbell helped with the literature review that supported our “Further readings and notes” sections. Time spent at the University of British Columbia made possible many stimulating conversations with Nancy Heckman and her colleagues. A discussion of many issues with Alois Kneip as well his hospitality for the first author at the University of Mainz was invaluable. The opportunity for us to spend time together afforded by St Peter’s College and the Department of Statistics at Oxford University was essential to the project.

April 2005

Jim Ramsay & Bernard Silverman

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