Multivariate Spline Functions and Their Applications

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Preface

As is known, the book named "Multivariate spline functions and their applications" has been published by the Science Press in 1994.

This book is an English edition based on the original book mentioned above with many changes, including that of the structure of a cubic C^1-interpolation in n-dimensional spline spaces, and more detail on triangulations have been added in this book.

Special cases of multivariate spline functions (such as step functions, polygonal functions, and piecewise polynomials) have been examined mathematically for a long time. I. J. Schoenberg (*Contribution to the problem of application of equidistant data by analytic functions, Quart. Appl. Math.*, 4(1946), 45 – 99; 112 – 141) and W. Quade & L. Collatz (*Zur Interpolations theories der reellen periodischen function, Press. Akad. Wiss.* (*PhysMath. KL*), 30(1938), 383 – 429) systematically established the theory of the spline functions. W. Quade & L. Collatz mainly discussed the periodic functions, while I. J. Schoenberg's work was systematic and complete. I. J. Schoenberg outlined three viewpoints for studing univariate splines: Fourier transformations, truncated polynomials and Taylor expansions. Based on the first two viewpoints, I. J. Schoenberg deduced the B-spline function and its basic properties, especially the basis functions. Based on the latter viewpoint, he represented the spline functions in terms of truncated polynomials. These viewpoints and methods had significantly effected on the development of the spline functions.

In view of the variety and complexity in application, it is very important to study the multivariate spline function theoretically. Since the multivariate spline function is heavily dependent on the geometric property of the domain partitions, it is so complex that the multivariate spline function, especially the non-Cartesian product multivariate spline function, has not been developed radically for a long time. G. Birkhoff, H. L. Garabedian, C. de Boor, M. H. Schultz and R. S. Varga discussed the Cartesian product bicubic spline function and its applications in numerical solutions of partial differential equations.

Analysing the relation between the polynomials over two adjacent cells, we introduce the smooth cofactor and conformality condition to which the polynomials must satisfy. The conformality condition establishes the equivalent conversion between the multivariate spline function and the corresponding algebraic problem. Moreover, the conformality condition provides an algebraic approach to studying the multivariate spline function. Based on the conformality condition theory, we have systematically studied the dimension of the multivariate spline functions, the basis functions, especially the locally supported basis functions, the smooth surface interpolations, the non-linear spline interpolations, the higher-dimensional spline functions, and the multivariate spline functions in computer aided geometric designs.

This book will systematically introduce the basic theories and methods on the multivariate spline functions. In order for the reader to know the frontier research on the multivariate spline functions, we will also introduce the modern developments of the multivariate spline functions and their applications in sciences and engineering. More precisely, Chapter 1 introduces the basic definitions of the multivariate spline functions, facts, and results; Chapter 2 mainly introduces the dimension of the multivariate spline function space the theory on the basis functions, and their constructions; Chapter 3 mainly introduces the notable Box spline, the simplex spline, and the B-net method, etc.; Chapter 4 introduces the basic theory, methods, and structures of the higher-dimensional spline functions; Chapter 5 introduces the theory on non-linear spline interpolations and their constructive methods; Chapter 6 introduces the basic problems and results on the piecewise algebraic curves and the piecewise algebraic surfaces; Chapter 7 introduces applications of the multivariate spline functions in the sciences and engineering, especially in finite element methods and computer aided geometric designs.

The writing of this book was participated in by professors Xiquan Shi, Zhongxuan Luo, Zhixun Su, and Dr. Shao-Ming Wang who is also the translator of this book. I wish to express my great appreciation to the Publishing Foundation of Academia Sinica, as will as The National

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Contents

Chapter 1

Introduction to Multivariate Spline Functions

It is well known that spline functions play very important roles in both theories and applications in the sciences and engineering. In view of the variety and complexity of the objectives, it is important to study the multivariate splines. Between the 1960's and early the 1970's, G. Birkhoff, H. L. Garabedian and Carl de Boor studied and established a series of theories on Cartesian tensor product multivariate splines. Although the Cartesian tensor product multivariate spline has its own application value, they are a simple extension of univariate spline functions, so they have many limitations.

In 1975, the author established a new approach to studying the basic theory of multivariate spline functions using the methods of function theory and algebraic geometry, and presented the so-called of smooth cofactor conformality method. Making use of this method, any problem on multivariate spline functions can be studied by transferring it into an equivalent algebraic problem.

Let D be a two dimensional domain in R^2, $\mathbf{P}_k$ be the collection of all these bivariate polynomials with real coefficients and total degree $\leq k$:

$$\mathbf{P}_k := \{\, p = \sum_{i=0}^{k}\sum_{j=0}^{k-i} c_{ij}x^iy^j \mid c_{ij} \text{ is a real value}\}.$$

A bivariate polynomial $p \in \mathbf{P}_k$ is called an irreducible polynomial if the polynomial can not be exactly divided by any other polynomial except a constant or itself (in the complex field). An algebraic curve

$$\Gamma : l(x, y) = 0, \quad l(x, y) \in \mathbf{P}_m$$

is called an irreducible algebraic curve if $l(x, y)$ is an irreducible polynomial. Clearly, a straight line is an irreducible algebraic curve.

Using a finite number of irreducible algebraic curves to carry out the partition Δ in a domain D, the domain D is divided into a finite number of sub-domains $D_1, D_2, \cdots, D_N$ by the partition Δ; each of such sub-domains is called a 'cell'. These line segments that form the boundary of each cell are called the 'mesh segments' (edge); intersection points of the mesh segments are called the 'mesh points' (vertex). The interior of a mesh segment has no mesh point, that is, only the two ends of the mesh segments are mesh points. All mesh points in a closed cell are called the vertices of this cell. If two mesh points are two end points of a single mesh segment, then these two mesh points are called adjacent mesh points.

Carrying out the partition Δ in the domain D, the union of all the cells with a certain mesh point V as a vertex is called an incidence domain or a star shape domain of the mesh point V relative to the partition Δ, denoted by $St(V)$.

The space of multivariate spline functions is defined by

$$S_k^\mu(\Delta) := \{s \in C^\mu(D) \mid \Phi \mid_{D_i} \in \mathbf{P}_k, i = 1, \cdots, N\}.$$

In fact, $s \in S_k^\mu(\Delta)$ is a piecewise polynomial of degree k possesses μ order continuous partial derivatives in D.

1.1 Basic frame of multivariate spline functions

In order to establish the basic frame of multivariate spline functions, we need the following lemmas.

Lemma 1.1 [1] *Let $p(x, y) \in \mathbf{P}_k$. If certain n zeros $(x_i, y_i)(i = 1, \cdots, n)$, $n \geq k + 1$ of a linear polynomial*

$$l(x, y) = ax + by + c, \qquad a^2 + b^2 \neq 0$$

are also the zeros of $p(x,y)$. Then $p(x,y)$ is exactly dividable by $l(x,y)$, that is, there is a polynomial $q(x,y) \in \mathbf{P}_{k-1}$ such that

$$p(x,y) = l(x,y) \cdot q(x,y), \tag{1.1}$$

Proof. Because a, b are not all zeroes, therefore we may assume that $b \neq 0$. Arranging $p(x,y)$ according to the order of descending power of y

$$p(x,y) = a_0(x)y^k + a_1(x)y^{k-1} + \cdots + a_{k-1}(x)y + a_k(x)$$

where $a_j(x),\ j = 0, \cdots, k$ is a polynomial of degree j of x. Divide by $l(x,y)$, we obtain

$$p(x,y) = l(x,y) \cdot q(x,y) + r(x), \tag{1.2}$$

where $q(x,y) \in \mathbf{P}_{k-1}$, the remainder $r(x)$ is a polynomial of degree not exceeding k. According to the assumption condition of the Lemma, we have

$$r(x_i) = 0, \quad i = 1, \cdots, n; \quad n \geq k+1. \tag{1.3}$$

Since $b \neq 0$, $x_i \neq x_j$ $(i \neq j)$. Therefore (1.3) shows that $r(x)$ has more than k zeros, and $r(x) = 0$. The Lemma is proved. □

Lemma 1.2 [1] *Let $p(x,y) \in \mathbf{P}_k$, and $q(x,y)$ be an irreducible algebraic polynomial. If $p(x,y)$ and $q(x,y)$ have more than km common zeros, then $p(x,y)$ is divisible by $q(x,y)$. Namely, there is a $r(x,y) \in \mathbf{P}_{k-m}$ such that $p(x,y) = q(x,y) \cdot r(x,y)$.*

According to Bezout's theorem in algebraic geometry, as long as $p(x,y)$ and $q(x,y)$ have more than km common zeroes, then they must have a common factor. Since $q(x,y)$ is irreducible, therefore $q(x,y)$ must be a factor of $p(x,y)$.

Theorem 1.3 [1] *Let the representation of $z = s(x,y)$ on the two arbitrary adjacent cells D_i and D_j be*

$$z = p_i(x,y) \text{ and } z = p_j(x,y)$$

where $p_i(x,y), p_j(x,y) \in \mathbf{P}_k$. In order to let $s(x,y) \in C^\mu(\overline{D_i \cup D_j})$, if and only if there is a polynomial $q_{ij}(x,y) \in \mathbf{P}_{k-(\mu+1)d}$ such that

$$p_i(x,y) - p_j(x,y) = [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y), \tag{1.4}$$

where $\overline{D_i}$ and $\overline{D_j}$ have common interior edge

$$\Gamma_{ij} : l_{ij}(x,y) = 0, \tag{1.5}$$

and the irreducible algebraic polynomial $l_{ij}(x,y) \in \mathbf{P}_d$.

Proof. Let μ be a given positive integer, $0 \le \mu \le k \cdot d^{-1} - 1$. According to the given condition, $s(x,y)$ is continuous everywhere in Γ_{ij}. Hence, $\eta(x,y) = p_i(x,y) - p_j(x,y)$ is equal to zero everywhere in Γ_{ij}. By Lemma 1.2, there is a polynomial $q_1(x,y) \in \mathbf{P}_{k-d}$, such that

$$\eta(x,y) = p_i(x,y) - p_j(x,y) = l_{ij}(x,y) \cdot q_1(x,y). \tag{1.6}$$

Also according to the property that the partial derivative of the first order of $\eta(x,y)$ being zero in Γ_{ij}, we know that

$$\begin{aligned} \frac{\partial q_1}{\partial x} l_{ij}(x,y) + q_1(x,y) \cdot \frac{\partial l_{ij}}{\partial x} |_{\Gamma_{ij}} = 0, \\ \frac{\partial q_1}{\partial y} l_{ij}(x,y) + q_1(x,y) \cdot \frac{\partial l_{ij}}{\partial y} |_{\Gamma_{ij}} = 0. \end{aligned} \tag{1.7}$$

Since $l_{ij}(x,y)$ is irreducible, by two equations in (1.7), we know that $q_1(x,y)$ is equal to zero everywhere in Γ_{ij}. Again make use of Lemma 1.2, there is a polynomial $q_2(x,y) \in \mathbf{P}_{k-2d}$ such that

$$q_1(x,y) = l_{ij}(x,y) \cdot q_2(x,y). \tag{1.8}$$

Then

$$\eta(x,y) = p_i(x,y) - p_j(x,y) = [l_{ij}(x,y)]^2 \cdot q_1(x,y). \tag{1.9}$$

Making use of the continuity of the partial derivative of the second order, the third order and up to μ order of $s(x,y) \in \overline{D_i \cup D_j}$, we obtain

$$\eta(x,y) = p_i(x,y) - p_j(x,y) = [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y), \tag{1.10}$$

where $q_{ij}(x,y) \in \mathbf{P}_{k-(\mu+1)d}$. □

The polynomial factor $q_{ij}(x,y)$ defined by (1.4) in Theorem 1.3 is called a *smooth cofactor* (cf.[1]) with interior mesh segment $\Gamma_{ij} : l_{ij}(x,y) = 0$ (from D_i to D_j). That means the existence of the smooth cofactor of interior mesh segment Γ_{ij} implies that the equality (1.4) is held.

By Theorem 1.3, we obtain the following corollary.

Corollary 1.4 [1] *Let mesh segments of the partition Δ be irreducible algebraic curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$. Their degrees are $n_1, n_2, \cdots, n_m$ respectively. Then in order for a surface $s \in S_k^\mu(\Delta)$ to exist (indeed piecewise), k and μ must satisfy the relation*

$$k \geq (\mu + 1) \cdot \min_i n_i. \tag{1.11}$$

Theorem 1.3 indicates that the multivariate spline function $s(x, y) \in S_k^\mu(\Delta)$ has a so-called the semi-analytic extension property, that is, the difference between two adjacent cells is only a modified term similar to the right hand side of (1.4). However, Theorem 1.3 could not completely represent the inner properties of multivariate spline functions. In order to provide the complete theoretic frame of multivariate spline functions, we need to do further research.

Let $\Gamma_{ij} : \pm l_{ij}(x, y) = 0$ be the common interior mesh segment between two adjacent cells D_i and D_j. Although the equation of Γ_{ij} can be both $l_{ij}(x, y) = 0$ and $-l_{ij}(x, y) = 0$, for the convenience and simplicity, we use only one form in the whole procession of discussion. We also assume that

$$\Gamma_{ij} = \Gamma_{ji}; \quad l_{ij}(x, y) = l_{ji}(x, y). \tag{1.12}$$

By (1.4), the smooth cofactor $q_{ij}(x, y)$ in Γ_{ij}, and the smooth cofactor $q_{ji}(x, y)$ in Γ_{ji} satisfy the relationship

$$q_{ij}(x, y) \equiv -q_{ji}(x, y). \tag{1.13}$$

Let A be a given interior mesh point. We adjust all the interior mesh line Γ_{ij} related to i and j which passing through A as follows: centered at A, crossing the mesh segment Γ_{ij} counter-clockwise, the moving point (x, y) just crosses from D_j to D_i.

Let A be an interior mesh point and define the *Conformality Condition* at A by

$$\sum_A [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \equiv 0, \tag{1.14}$$

where $\sum\limits_A$ presents the summation of all the interior mesh segments around A, and $q_{ij}(x, y)$ is the smooth cofactor on Γ_{ij}.

Let $A_1, \cdots, A_M$ be all the interior mesh points in Δ. The *Global Conformality Condition* is

$$\sum_{A_v} [l_{ij}(x, y)]^{\mu+1} \cdot q_{ij}(x, y) \equiv 0, \quad v = 1, \cdots, M, \tag{1.15}$$

where $q_{ij}(x, y)$ satisfies (1.14) the conformality condition corresponding to A_v.

The following theorem set up the basic frame of multivariate spline theory.

Theorem 1.5 [1] *Let* Δ *be any partition of* D. *The multivariate spline function* $s(x, y) \in S_k^\mu(\Delta)$ *exists, if and only if for every interior mesh segment, there exists a smooth cofactor of the* $s(x, y)$, *and satisfies the global conformality condition* (1.15).

In fact, the existence of the smooth cofactor on every interior mesh segment is equivalent to the C^μ smooth continuity of the piecewise polynomial. The property of conformality condition being satisfied at every interior mesh line, that is, the satisfaction of global conformality condition, is also equivalent to the single-valued property on the whole domain of the piecewise polynomial. Therefore Theorem 1.5 is true. The reader may write out some details (cf.[1]).

If spline function $s(x, y) \in S_k^\mu(\Delta)$ is a polynomial of degree k everywhere in a related domain $St(V)$ at some mesh point V, then we call $s(x, y)$ is degenerate over $St(V)$. If $s(x, y)$ is a polynomial of degree k over all the cells, then we say it is global degenerate. According to Theorem 1.5, $s(x, y)$ is degenerate over $St(V)$ means that there is only a zero solution of the corresponding conformality condition (1.14) at mesh point V. The global degeneracy means that there is only a zero solution of the global conformality condition (1.15).

In view of the purpose of multivariate spline function being to study some theories and practical problems, we are interested in how to select the partition Δ, the degree k and the smoothness μ such that there exists a non-degenerate multivariate spline function. Theorem 1.5 shows that there is a radical difference between multivariate spline function and univariate spline function. The relations of the domain D, the partition Δ, the degree k of piecewise polynomial and the smoothness μ, that is the effect of global conformality condition (1.15), eventually determines a multivariate spline function. In fact, Theorem 1.5 points out that the multivariate spline function is equivalent to the linear algebraic problem of (1.15): the problem of homogeneous system of linear equations on the coefficients of the smooth cofactors. The existence of solution and its properties of this kind of homogeneous system of linear equations become the key issue to study the multivariate spline functions.

Suppose that the boundary ∂D of domain D is composed of some irreducible algebraic curves. If these irreducible algebraic curves are a part of the interior mesh segments on the whole plane R^2, and yield a partition $\overline{\Delta}$ of R^2 with the original partition Δ of D, then it is called a global partition, while $R^2 \backslash D$ is also a cell of $\overline{\Delta}$.

As a direct corollary of Theorem 1.5, we have (cf.[1])

Corollary 1.6 *For a global partition $\overline{\Delta}$, there is a $s(x,y) \in S_k^\mu(\overline{\Delta})$, if and only if for every interior mesh segment, there exists a smooth cofactor of the $s(x,y)$, and satisfies the global conformality condition* (1.15) *at every mesh point.*

Obviously, conformality condition ensures the single-valued property of $s(x,y)$ over Δ and $\overline{\Delta}$. If domain is not simply connected, for instance, D is a multi-connected domain with h number of holes, then Theorem 1.5 and Corollary 1.6 are still true with an additional **Hole Conformality Condition**

$$\sum_{H_r} [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y) \equiv 0, \ r = 1, \cdots, h \tag{1.16}$$

where $\sum\limits_{H_r}$ presents the summation of all the interior mesh segments across the rth hole. The other notation in (1.16) is the same as (1.14) and (1.15).

In R^2, every straight line $\Gamma : l(x,y) \equiv ax + by + c = 0$ is obviously an irreducible algebraic curve. Therefore, for the partition in which all the mesh lines are straight, the above results are still true. For instance,

Theorem 1.7 [1] *Let the representation of $z = s(x,y)$ on the two adjacent cells D_i and D_j be $z = p_i(x,y)$ and $z = p_j(x,y)$ respectively. In order to let $s(x,y) \in C^\mu(\overline{D_i \cup D_j})$, if and only if there exists a polynomial $q_{ij}(x,y) \in \mathbf{P}_{k-(\mu+1)d}$ such that*

$$p_i(x,y) - p_j(x,y) = [l_{ij}(x,y)]^{\mu+1} \cdot q_{ij}(x,y), \tag{1.17}$$

where $\Gamma_{ij} : l_{ij}(x,y) \equiv a_{ij}x + b_{ij}y + c_{ij} = 0$ is the common interior edge of D_i and D_j.

Theorem 1.8 [1] *Let Δ be a straight line partition of D. The multivariate spline function $s(x,y) \in S_k^\mu(\Delta)$ exists, if and only if for every interior mesh segment, there exists a smooth cofactor of the $s(x,y)$, and satisfies the global conformality condition*

$$\sum_{A_v} [l_i(x,y)]^{\mu+1} \cdot q_i(x,y) \equiv 0, \tag{1.18}$$

where A_v goes through all the interior mesh points, $l_i(x,y) \equiv a_i x + b_i y + c_i = 0$ is an interior mesh segment around A_v, and $q_i(x,y) \in \mathbf{P}_{k-\mu-1}$ is the smooth cofactor on $l_i(x,y)$.

Propositions in paper [1] show that if there is any constraint condition on the boundary ∂D of D and we expect bivariate spline function $s(x,y) \in C^1$ over arbitrary triangulation, the degree of the piecewise polynomial should not be less than five unless we select a special triangulation.

If we set up a partition Δ of D as follows: all the mesh segments are straight lines cross-cut domain D. This kind of partition is called a **Cross-cut partition**. In view of the speciality of cross-cut partition, we have:

Theorem 1.9 [1] *If partition Δ is a cross-cut partition, then there is always a non-degenerate multivariate spline function $s(x,y) \in S_k^\mu(\Delta)$, $k \geq \mu + 1$.*

Let $\Gamma_i : a_i x + b_i y + c_i = 0$ be an arbitrary mesh segment of partition Δ. We define

$$\Gamma_i^- = \{(x,y) \in D \mid a_i x + b_i y + c_i < 0\},$$

$$\Gamma_i^+ = \{(x,y) \in D \mid a_i x + b_i y + c_i > 0\}.$$

For any mesh segment Γ_{ij} derived from Γ_i which $\Gamma_{ij} \subset \Gamma_i$, if we let non-zero polynomial $q_i(x,y) \in \mathbf{P}_{k-\mu-1}$ be the smooth cofactor of Γ_i from Γ_i^- to Γ_i^+, then it is easy to see that the corresponding global conformality condition must be satisfied. Therefore, Theorem 1.9 is true. Especially, we have

Corollary 1.10 [1] *If partition Δ is a rectangular partition, then there is always a non-degenerate multivariate spline function $s(x,y) \in S_k^\mu(\Delta)$, $k \geq \mu + 1$.*