INEQUALTIES

不等式

Second Edition

第2版

G. Hardy, J. E. Littlewood & G. Pólya

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世界图书出版公司

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011-4211, USA www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia
Ruiz de Alarcón 13, 28014 Madrid, Spain

© Cambridge University Press 1934, 1952

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First published 1934
Second edition 1952
Reprinted 1959, 1964, 1967, 1973, 1988 (twice)
First paperback edition 1988
Reprinted 1989, 1991, 1994, 1997, 1999

A catalogue record for this book is available from the British Library

ISBN 0 521 05206 8 hardback
ISBN 0 521 35880 9 paperback

This edition of Inequalities 2nd ed by G.Hardy , J.E.Littlewood & G.Pólya is published by arrangement with the Syndicate of the Press of University of Cambridge, Cambridge, England.

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PREFACE TO FIRST EDITION

This book was planned and begun in 1929. Our original intention was that it should be one of the *Cambridge Tracts*, but it soon became plain that a tract would be much too short for our purpose.

Our objects in writing the book are explained sufficiently in the introductory chapter, but we add a note here about history and bibliography. Historical and bibliographical questions are particularly troublesome in a subject like this, which has applications in every part of mathematics but has never been developed systematically.

It is often really difficult to trace the origin of a familiar inequality. It is quite likely to occur first as an auxiliary proposition, often without explicit statement, in a memoir on geometry or astronomy; it may have been rediscovered, many years later, by half a dozen different authors; and no accessible statement of it may be quite complete. We have almost always found, even with the most famous inequalities, that we have a little new to add.

We have done our best to be accurate and have given all references we can, but we have never undertaken systematic bibliographical research. We follow the common practice, when a particular inequality is habitually associated with a particular mathematician's name; we speak of the inequalities of Schwarz, Hölder, and Jensen, though all these inequalities can be traced further back; and we do not enumerate explicitly all the minor additions which are necessary for absolute completeness.

We have received a great deal of assistance from friends. Messrs G. A. Bliss, L. S. Bosanquet, R. Courant, B. Jessen, V. Levin, R. Rado, I. Schur, L. C. Young, and A. Zygmund have all helped us with criticisms or original contributions. Dr Bosanquet, Dr Jessen, and Prof. Zygmund have read the

proofs, and corrected many inaccuracies. In particular, Chapter III has been very largely rewritten as the result of Dr Jessen's suggestions. We hope that the book may now be reasonably free from error, in spite of the mass of detail which it contains.

Dr Levin composed the bibliography. This contains all the books and memoirs which are referred to in the text, directly or by implication, but does not go beyond them.

G. H. H.
J. E. L.
G. P.

Cambridge and Zürich
July 1934

PREFACE TO SECOND EDITION

The text of the first edition is reprinted with a few minor changes; three appendices are added.

J. E. L.
G. P.

Cambridge and Stanford
March 1951

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CHAPTER I

INTRODUCTION

1.1. Finite, infinite, and integral inequalities. It will be convenient to take some particular and typical inequality as a text for the general remarks which occupy this chapter; and we select a remarkable theorem due to Cauchy and usually known as 'Cauchy's inequality'.

Cauchy's inequality (Theorem 7) is

$$(1.1.1)\qquad (a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2 \leqq (a_1{}^2 + a_2{}^2 + \ldots + a_n{}^2)(b_1{}^2 + b_2{}^2 + \ldots + b_n{}^2)$$

or

$$(1.1.2)\qquad (\sum_1^n a_\nu b_\nu)^2 \leqq \sum_1^n a_\nu{}^2 \sum_1^n b_\nu{}^2,$$

and is true for all real values of $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$. We call $a_1, \ldots, b_1, \ldots$ the *variables* of the inequality. Here the number of variables is finite, and the inequality states a relation between certain finite sums. We call such an inequality an *elementary* or *finite* inequality.

The most fundamental inequalities are finite, but we shall also be concerned with inequalities which are not finite and involve generalisations of the notion of a sum. The most important of such generalisations are the infinite sums

$$(1.1.3)\qquad \sum_1^\infty a_\nu, \quad \sum_{-\infty}^\infty a_\nu$$

and the integral

$$(1.1.4)\qquad \int_a^b f(x)\,dx$$

(where a and b may be finite or infinite). The analogues of (1.1.2) corresponding to these generalisations are

$$(1.1.5)\qquad (\sum_1^\infty a_\nu b_\nu)^2 \leqq \sum_1^\infty a_\nu{}^2 \sum_1^\infty b_\nu{}^2$$

(or the similar formula in which both limits of summation are infinite), and

$$\left(\int_a^b f(x)\,g(x)\,dx\right)^2 \leqq \int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx. \tag{1.1.6}$$

We call (1.1.5) an *infinite*, and (1.1.6) an *integral*, inequality.

1.2. Notations. We have often to distinguish between different *sets* of the variables. Thus in (1.1.2) we distinguish the two sets $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$. It is convenient to have a shorter notation for sets of variables, and often, instead of writing 'the set $a_1, a_2, \ldots, a_n$' we shall write 'the set (a)' or simply 'the a'.

We shall habitually drop suffixes and limits in summations, when there is no risk of ambiguity. Thus we shall write

$$\Sigma a$$

for any of

$$\sum_1^n a_\nu, \quad \sum_1^\infty a_\nu, \quad \sum_{-\infty}^\infty a_\nu;$$

so that, for example,

$$(\Sigma ab)^2 \leqq \Sigma a^2 \, \Sigma b^2 \tag{1.2.1}$$

may mean either of (1.1.2) or (1.1.5), according to the context.

In integral inequalities, the *set* is replaced by a *function*; thus in passing from (1.1.2) to (1.1.6), (a) and (b) are replaced by f and g. We shall also often omit variables and limits in integrals, writing

$$\int f dx$$

for (1.1.4): so that (1.1.6), for example, will be written as

$$\left(\int fg\,dx\right)^2 \leqq \int f^2 dx \int g^2 dx. \tag{1.2.2}$$

The ranges of the variables, whether in sums or integrals, are prescribed at the beginnings of chapters or sections, or may be inferred unambiguously from the context.

1.3. Positive inequalities. We are interested primarily in 'positive' inequalities[a]. A finite or infinite inequality is *positive* if all variables $a, b, \ldots$ involved in it are real and non-negative. An inequality of this type usually carries with it, as a trivial

[a] There are exceptions, as for example in §§ 8.8–8.17. There the 'positive' cases of the theorems discussed are relatively trivial.

corollary, an apparently more general inequality valid for all real, or even complex, $a, b, \ldots$. Thus from (1.1.2) and the inequality

$$|\Sigma u| \leqq \Sigma |u|, \tag{1.3.1}$$

valid for all real or complex u, we deduce

$$|\Sigma ab|^2 \leqq (\Sigma |a||b|)^2 \leqq \Sigma |a|^2 \Sigma |b|^2, \tag{1.3.2}$$

where the a and b are arbitrary complex numbers. We shall usually be content to state our theorems in the fundamental 'positive' form and to leave the derived results to the reader. Occasionally, however, when the inequality in question is very important, we state it explicitly in its most general form.

Similar remarks apply to integral inequalities. The independent variable x will be real, but will (like the variable of summation ν) take either positive or negative values; while the functions $f(x)$, $g(x)$, ... will generally assume non-negative values only. To such an inequality as (1.1.6), true for non-negative f, g, corresponds the more general inequality

$$\left|\int fg\,dx\right|^2 \leqq \int |f|^2 dx \int |g|^2 dx, \tag{1.3.3}$$

valid for arbitrary complex functions f, g of the real variable x.

Numbers $k, l, r, s, \ldots$ occurring as indices in our theorems are real but in general capable of either sign.

1.4. Homogeneous inequalities. The two sides of (1.1.2) are homogeneous functions of degree 2 of the a and also of the b; and generally both sides of our inequalities will be homogeneous functions, of the same degree, of certain sets of variables. Since homogeneous functions of positive degree vanish when all their arguments vanish, both sides, if of positive degree, will vanish, and so be equal, when the sets concerned consist entirely of 0's. Thus (1.1.2) reduces to an equality if all the a, or all the b, are 0.

A set consisting entirely of 0's is called a *null set*, or *the null set*, if the context is unambiguous. In general the '$\leqq$' or '$\geqq$' of our theorems will reduce to '$=$' when one or all of the sets involved is null. Sometimes this will be the *only* case of equality. More usually there will be other cases; thus plainly '$=$' occurs in (1.1.2) if every a is equal to the corresponding b. We shall be careful, wherever it is possible, to pick out explicitly such cases of equality.

The homogeneity of an inequality in certain sets of variables often enables us to simplify our proofs by imposing an additional restriction (a *normalisation*) on them. Thus the means $\mathfrak{M}_r(a)$ of § 2.2 are homogeneous, of degree 0, in the weights p, and we may always suppose, if we please, that $\Sigma p = 1$. Again, if we wish to prove that

$$(1.4.1) \qquad (a_1{}^s + a_2{}^s + \dots + a_n{}^s)^{1/s} \leqq (a_1{}^r + a_2{}^r + \dots + a_n{}^r)^{1/r}$$

when $0 < r < s$ (Theorem **19**), we may suppose (since both sides are homogeneous in the a of degree 1) that $\Sigma a^r = 1$. We have then

$$a_\nu{}^r \leqq 1, \quad a_\nu{}^s = (a_\nu{}^r)^{s/r} \leqq a_\nu{}^r,$$

and so $\Sigma a^s \leqq \Sigma a^r = 1$. Without this preliminary normalisation, our proof would run

$$\frac{(\Sigma a^s)^{1/s}}{(\Sigma a^r)^{1/r}} = \left\{\Sigma \frac{a^s}{(\Sigma a^r)^{s/r}}\right\}^{1/s} = \left\{\Sigma \left(\frac{a^r}{\Sigma a^r}\right)^{s/r}\right\}^{1/s} \leqq \left(\Sigma \frac{a^r}{\Sigma a^r}\right)^{1/s} = 1.$$

There is another sense of 'homogeneity' which is sometimes important. Let us compare (1.4.1) above, which may be written as

$$(1.4.2) \qquad (\Sigma a^s)^{1/s} \leqq (\Sigma a^r)^{1/r},$$

with (1.1.2). Both inequalities are homogeneous in the variables, but (1.1.2) has a further homogeneity which (1.4.2) has not. It is, as we may say, 'homogeneous in Σ'; Σ, if treated as a number, would occur to the same power on the two sides of the inequality.

The result of this homogeneity in Σ is that (1.1.2) remains true if every *sum* which occurs is replaced by the corresponding *mean*, i.e. if written in the form

$$\left(\frac{1}{n}\Sigma ab\right)^2 \leqq \left(\frac{1}{n}\Sigma a^2\right)\left(\frac{1}{n}\Sigma b^2\right).$$

The importance of this kind of homogeneity will appear very clearly in § 2.10 and § 6.4. Roughly, an inequality which possesses it has an integral analogue, while one which does not, like (1.4.2), has none.

1.5. The axiomatic basis of algebraic inequalities[a]. Our subject is difficult to define precisely, but belongs partly to 'algebra' and partly to 'analysis'. Algebra or analysis, like geometry, may be treated axiomatically. Instead of saying, as

[a] See Artin and Schreier (1).

for example in Dedekind's theory of real numbers, that we are concerned with such or such definite objects, we may say, as in projective geometry, that we are concerned with any system of objects which possesses certain properties specified in a set of axioms. We do not propose to consider the 'axiomatics' of different parts of the subject in detail, but it may be worth while to insert a few remarks concerning the axiomatic basis of those theorems which, like (1.1.2) and most of the theorems of Ch. II, belong properly to algebra.

We may take as the axioms of an algebra only the ordinary laws of addition and multiplication. All our theorems will then be true in many different fields, in real algebra, complex algebra, or the arithmetic of residues to any modulus. Or we may add axioms concerning the solubility of linear equations, axioms which secure the existence and uniqueness of difference and quotient. Our theorems will then be true in real or complex algebra or in arithmetic to a *prime* modulus.

In our present subject we are concerned with relations of *inequality*, a notion peculiar to *real* algebra. We can secure an axiomatic basis for theorems of inequality by taking, in addition to the 'indefinables' and axioms already referred to, one new indefinable and two new axioms. We take as indefinable the idea of a *positive* number, and as axioms the two propositions:

I. *Either a is* 0 *or a is positive or $-a$ is positive, and these possibilities are exclusive.*

II. *The sum and product of two positive numbers are positive.*

We say that a is *negative* if $-a$ is positive, and that a is *greater* (*less*) than b if $a-b$ is positive (negative). Any inequality of a purely algebraic type, such as (1.1.2), may be made to rest on this foundation.

1.6. Comparable functions. We may say that the functions

$$f(a)=f(a_1, a_2, \ldots, a_n), \quad g(a)=g(a_1, a_2, \ldots, a_n)$$

are *comparable* if there is an inequality between them valid for all non-negative real a, that is to say if either $f \leqq g$ for all such a or

$f \geqq g$ for all such a. Two given functions are not usually comparable. Thus two positive homogeneous polynomials of different degrees are certainly not comparable[a]; if $0 \leqq f \leqq g$ for all non-negative a, and both sides are homogeneous, then f and g are certainly of the same degree.

The definition may naturally be extended to functions $f(a, b, \ldots)$ of several sets of variables.

We shall be occupied throughout this volume with problems concerning the comparability of functions. Thus the arithmetic and geometric means of the a are comparable: $\mathfrak{G}(a) \leqq \mathfrak{A}(a)$ (Theorem **9**). The functions $\mathfrak{G}(a+b)$ and $\mathfrak{G}(a)+\mathfrak{G}(b)$ are comparable (Theorem **10**). The functions $\mathfrak{A}(ab)$ and $\mathfrak{A}(a)\,\mathfrak{A}(b)$ are not comparable; their relative magnitude depends upon the relations of magnitude of the a and b (Theorem **43**). The functions

$$\psi^{-1}(\Sigma p \psi(a)), \quad \chi^{-1}(\Sigma p \chi(a))$$

are comparable if and only if $\chi\psi^{-1}$ is convex or concave (Theorem **85**).

An important general theorem concerning the comparability of two functions of the form

$$\Sigma a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_n^{\alpha_n},$$

due to Muirhead, will be found in § 2.18.

1.7. Selection of proofs. The methods of proof which we use in different parts of the book will depend on very different sets of ideas, and we shall often, particularly in Ch. II, give a number of alternative proofs of the same theorem. It may be useful to call attention here to certain broad distinctions between the methods which we employ.

In the first place, many of the proofs of Ch. II are 'strictly elementary', since they depend solely on the ideas and processes of finite algebra. We have made it a principle to give at any rate one such proof of any really important theorem whose character permits it.

Next we have, even in Ch. II, many proofs which are not elementary in this sense because they involve considerations of

[a] Compare § 2.19.

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