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TRIGONOMETRIC SERIES

R. L. JEFFERY

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A Survey by

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Mr. Chairman and Fellows of Section III of the Royal Society of Canada:

It was with apprehension that I looked forward to the time when, as President of this Section, I would be called upon to give an address. To prepare a mathematical talk for an audience that is predominantly non-mathematical is a heavy undertaking. Fortunately I have a side interest that often intrudes itself into your work. The Astronomer, the Chemist, the Engineer and the Physicist are continually meeting with trigonometric series as solutions of the differential equations that arise in the problems they are called upon to solve. In fact the origin of the study of trigonometric series was in the search for solutions of such equations, and no topic arising out of applied and experimental work has been so great a challenge to the professional mathematician or so great a stimulus to the advancement of pure mathematics. The main purpose of my address is to show the extent to which this is true. I shall go back to the beginning, describe the main problems that have arisen, and give some indication of the manner in which they have been solved. The time it has taken to settle the various stages will be emphasized as a measure of the difficulties encountered. To bring the subject up to the present in a general way is the purpose of the first part of the address. In the second part the proofs of the main-line theorems will be given.

PART I

Introduction. There are at least four separate discussions of the earlier phases of the subject **(1, 2, 3, 4)**. Consequently I shall take from this period only what I need to illustrate the points I wish to make.

1. The period of d'Alembert, Euler and Daniel Bernoulli. Most of you are familiar with the story of the vibrating string. Let an elastic string be stretched taut on the x-axis with ends at $(l, 0)$, $(-l, 0)$. If this string is displaced and released it vibrates in such a way that the ordinate of a point on the string is a function of the time t and the x-coordinate of the point, $y = y(t, x)$.

As early as 1747 the French mathematician d'Alembert knew that this function satisfied the differential equation

$$(1) \qquad \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

This in itself is remarkable when we consider that both Newton and Leibnitz who invented the Calculus were alive in the early 1700's. Between 1747 and 1753, d'Alembert, Euler and Bernoulli gave their attention to the solution of this equation and showed that it involved representing the initial position of the string at the time of release,

$$y(0, x) = f(x),$$

by a trigonometric series of the form

$$(2) \qquad \tfrac{1}{2} a_0 + \sum_{1}^{\infty} (a_k \cos k\pi x/l + b_k \sin k\pi x/l).$$

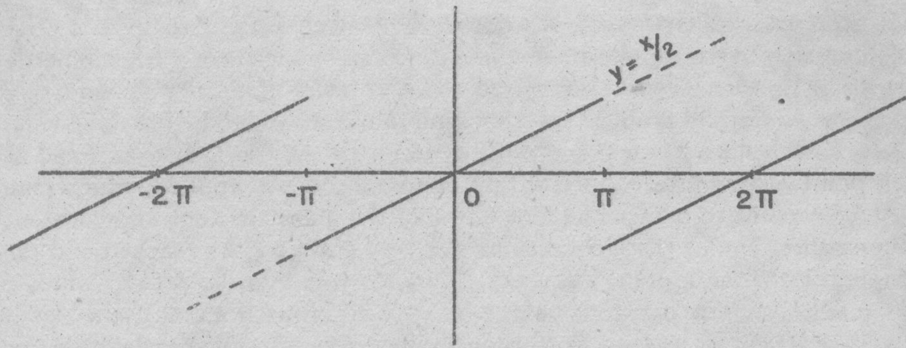
This posed two questions:

I. If $y(0, x)$ could be represented by such a series how could the coefficients a_k, b_k be determined?

II. It was clear that there was a considerable degree of arbitrariness in the way the string could be constrained in its initial position. For example $y(0, x)$ could be in part a straight line, in part an arc of a circle, in part a piece of a sine curve. Was it reasonable to expect that the single expression (2) could represent a straight line on part of the interval $(-l, l)$, a circle on another part of this interval, and a sine curve on still another part? To the mathematicians of that day this seemed absurd. In this point of view Euler was the most emphatic. He took his stand on the ground that the function (2) was periodic and could, therefore, represent only periodic functions. He would not accept as a possibility that a series such as (2) could represent a function in a certain interval and fail to represent the function outside that interval. It seems that Euler never receded from this position even though later he himself discovered such relations as

$$(3) \qquad \frac{x}{2} = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \dots \qquad -\pi < x < \pi.$$

The graph of the right side is, for all x, the periodic function in heavy line in the accompanying diagram. The series represents the line $y = x/2$ only for values of x on $-\pi < x < \pi$. But Euler saw in this and similar problems no reason to believe that if an arbitrary function was given the series (2) could be so determined that it would represent the function on an interval. It appears that d'Alembert held much the same view in regard to the possibility of representing an arbitrary function by means of a trigonometric series.



In 1755 Daniel Bernoulli published a memoir in which he maintained that the motion of the string $y(t, x)$ is expressible in the form

$$(4) \qquad y(t, x) = \sum_{1}^{\infty} A_k \sin \frac{k\pi x}{l} \cos \frac{k\pi a t}{l}$$

which, when $t = 0$, reduces to

$$(5) \qquad \sum_{1}^{\infty} A_k \sin \frac{k\pi x}{l}.$$

Bernoulli considered that the initial position of displacement of the string could always be represented by (5). Against this point of view d'Alembert and Euler brought the full force of their arguments as outlined above. Bernoulli's analysis had been sketchy, and in places more speculative than rigorous, which left his position weak. He was, however, unshaken by the objections of his opponents. He pointed out that an arbitrary function defined on $(-l, l)$ could be represented at a given finite set of m points by a finite number of terms of (5) if the coefficients were properly chosen. He, therefore, saw no reason for rejecting the possibility that the infinite series of the form (5) could coincide with the values of an arbitrary function at more than a finite number of points.

In this brief sketch we have shown the uncertainties under which the studies of trigonometric series began. A more complete story of these beginnings can be found in the references cited above. *What we wish to emphasize here is that the d'Alembert-Euler-Bernoulli controversy, with contributions from Lagrange, went on for more than a decade without in any way settling questions I and II.*

In 1777 Euler led the way in showing that if a function could be represented as in (2) then the coefficients must of necessity be of the form

$$(6) \qquad a_k = \frac{1}{l}\int_{-l}^{l} f(t) \cos \frac{k\pi t}{l}\, dt, \qquad b_k = \frac{1}{l}\int_{-l}^{l} f(t) \sin \frac{k\pi t}{l}\, dt.$$

This was, after thirty years, a partial answer to I.

2. The contribution of Fourier. The next person to give serious consideration to the representation of a function by means of a trigonometric series was Fourier. He was interested in the transfer of heat in a conducting medium. A simple problem of the kind he considered is the following: A long metal plate has one end on the interval $(-l, l)$ on the x-axis and at each point of this interval a constant source of heat is applied, which may vary from point to point. The long edges of the plate are kept at a constant temperature. Under these conditions a steady state will be reached and the temperature T at a point (x, y) on the plate will then be a function of x and y, $T(x, y)$. The constant source of heat distributed along the x-axis is $T(x, 0) = f(x)$. It was at that time known that the differential equation which this function $T(x, y)$ satisfies is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{7}$$

It was also known that a solution of this equation for a given set of boundary conditions involved representing the function $T(x, 0) = f(x)$ in the form (2). Fourier was a physicist. He wanted answers to applied problems. Whether or not the steps he took were logically justifiable was not his main concern. In 1807 he stated and used the following theorem.

Any single valued function $f(x)$ defined on $(-l, l)$ is represented over this interval by a series of sines and cosines in the manner

$$f(x) = \frac{a_0}{2} + \sum_1^\infty \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right)$$

where

$$a_k = \frac{1}{l} \int_{-l}^{l} f(t) \cos \frac{k\pi t}{l}\, dt, \qquad b_k = \frac{1}{l} \int_{-l}^{l} f(t) \sin \frac{k\pi t}{l}\, dt.$$

If the interval is $(0, l)$ then either sines or cosines suffice, the series being in the one case

$$f(x) = \sum_1^\infty b_k \sin \frac{k\pi x}{l}, \text{ with } b_k = \frac{2}{l} \int_0^l f(t) \sin \frac{k\pi t}{l}\, dt,$$

and in the other case

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_k \cos \frac{k\pi x}{l}, \text{ with } a_k = \frac{2}{l} \int_0^l f(t) \cos \frac{k\pi t}{l}\, dt.$$

Fourier placed no restrictions on the functions other than integrability which is implied by the definitions of the coefficients a_k, b_k. Fourier's statement was received with unbelief by those who had given serious thought to the logical aspects of the problem.

We shall see later that there were sound reasons for this unbelief. Fourier himself was not as complacent in his attitude as the above remarks imply. He did much in the way of verification of the formulas by working with special cases and showing that for values of n for which it was practicable to make the computations, the sum

$$\frac{a_0}{2} + \sum_1^n \left(a_k \cos \frac{k\pi x}{l} + b_k \sin \frac{k\pi x}{l} \right)$$

was close to the value of the function. This did not ensure that for larger values of n the approximation was as good or better.

In spite of his best efforts Fourier accomplished nothing in the way of logical proofs as to the class of functions that could be represented by a trigonometric series. Nevertheless, what he did in the way of investigating special cases and in the way of using trigonometric series in applied work was a valuable contribution in recognition of which trigonometric series associated with a function $f(x)$ *by means of the Euler formulas* (6) *have come to be called Fourier series.*

The end of Fourier's investigations is nearly a hundred years from the beginning of the study of trigonometric series. Yet the secrets of these series were still withholding themselves. Such men as Euler, d'Alembert, Lagrange, Bernoulli, walked on the very edge of these secrets without falling upon them. Probably this failure was in some measure caused by the narrow concept of a function which prevailed at that time. It was considered that a function is defined on an interval only if it is given by a single formula on the interval. To quote from Langer (**1**, p. 5) "A more conspicuous example of the confining effects of preconceptions is hardly to be found." That the work of the hundred years was so fruitless will appear all the more amazing when we see in Part II with what ease present-day methods get at most of the facts.

3. Considerations of notation. Each term of the series (2) is periodic with period $2l$. Hence if it represents a function $f(x)$ on $[-l, l]$ let this function be defined outside of $[-l, l]$ by periodicity with period $2l$. Then (2) represents $f(x)$ for all values of x. Set $\pi x/l = x'$. Then $f(x) = f(lx'/\pi) = F(x')$. Also

$$f(x) = f(x + 2l) = f\left[\frac{l(x' + 2\pi)}{\pi}\right] = F(x' + 2\pi) = F(x')$$

which shows that $F(x')$ is periodic with period 2π. Also, with this transformation $\pi t/l = t'$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t')\cos kt'dt', \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t')\sin kt'dt'.$$

Again, since the integrands in the formulas defining a_k, b_k are periodic with period 2π, the integration may be taken over any interval of length 2π. Then if the periodic function $f(x)$ is given by (2) with coefficients a_k, b_k determined by (6), the periodic function $F(x')$ is given by

$$F(x') = \frac{a_0}{2} + \sum_1^\infty (a_k \cos kx' + b_k \sin kx')$$

with a_k, b_k determined by

$$a_k = \frac{1}{\pi}\int_0^{2\pi} F(t')\cos kt'dt', \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} F(t')\sin kt'dt'.$$

It is thus seen that the study of a function on an arbitrary interval relative to its representation by a Fourier series can be reduced to the study of a related function on the interval $[0, 2\pi]$. Consequently it is customary to start the study of Fourier series with an integrable function $f(x)$ defined on $0 \leqslant x < 2\pi$ and defined by periodicity with period 2π outside this interval.

As noted above, it has now become the custom to call

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\, dt, \qquad b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\, dt,$$

$$\frac{a_0}{2} + \sum_1^\infty (a_k \cos kx + b_k \sin kx)$$

respectively the Fourier coefficients and the Fourier series of the function $f(x)$. We shall adopt this practice in what follows.

The first positive result, which was indicated by Euler in 1777, was the following.

If $f(x)$ is an integrable function on $[0, 2\pi]$ and

$$f(x) = \frac{a_0}{2} + \sum_1^\infty (a_k \cos kx + b_k \sin kx)$$

then a_k and b_k must be the Fourier coefficients.

Multiply both sides by $\cos kx$ and integrate between 0 and 2π, to get

$$\int_0^{2\pi} f(x)\cos kx\, dx = \frac{a_0}{2}\int_0^{2\pi} \cos kx\, dx + a_1 \int_0^{2\pi} \cos x \cos kx\, dx + \ldots$$
$$+ a_k \int_0^{2\pi} \cos^2 kx\, dx + \ldots.$$

This procedure implies that the series obtained by multiplying by $\cos kx$ is integrable term by term. Because the first of the relations

$$\int_0^{2\pi} \cos mx \sin nx\, dx, \qquad \int_0^{2\pi} \cos mx \cos nx\, dx,$$

is zero for all m and n and the second is zero on π according as $m \neq n$; $m = n$; $m, n = 0, 1, 2, \ldots$, every term on the right is zero except the one involving a_k which is πa_k. If $k = 0$ every term on the right is zero except the first which is πa_0. Thus

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt, \qquad k = 0, 1, 2, \ldots.$$

We now see why the constant term of a Fourier series is written as $a_0/2$. Otherwise this formula for a_k would not fit when $k = 0$. Similarly it can be shown that

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt.$$

It follows from the foregoing that if an integrable function on $[0, 2\pi]$ is represented by a trigonometric series this series is the Fourier series of $f(x)$. But this throws no light whatever on the problem of determining whether or not a specific function can be represented by a trigonometric series.

4. The contributions of Dirichlet. The first to reveal some of the inner secrets of trigonometric series was the German mathematician Lejeune-Dirichlet. In 1829 he firmly established the following theorem.

If the function $f(x)$ defined on the interval $[0, 2\pi]$ has only a finite number of ordinary discontinuities and only a finite number of maxima and minima then, if a_k, b_k are the Fourier coefficients of $f(x)$,

$$S_n(x) = \frac{a_0}{2} + \sum_1^n (a_k\cos kx + b_k\sin kx)$$

tends to $f(x)$ at a value of x for which $f(x)$ is continuous, and $S_n(x)$ tends to $\{f(x+0) + f(x-0)\}/2$ at a point of jump.

Here at last was progress. True it did not show that a continuous function with an infinite number of maxima and minima could be represented by a Fourier series. Nevertheless, it was generally thought that this would come as an incidental result in the consolidating of gains already made, and interest waned.

In 1876 du Bois-Reymond dashed all hope of a complete conquest of the problem of representing a continuous function by a Fourier series. *He constructed a continuous function for which the Fourier representation failed at a single point. This led to an example where the representations failed at each of an everywhere dense set of points.*

5. Progress during the late 1800's and early 1900's. With du Bois-Reymond's discovery there was renewed interest, which led to the following result.

If $f(x)$ is of bounded variation on the closed interval $[0, 2\pi]$ then at every point x on $[0, 2\pi]$ the Fourier series converges to $\{f(x+0) + f(x-0)\}/2$.

This is only a slight advance over Dirichlet's results. If $f(x)$ is of bounded variation there is a number M such that for any set of non-overlapping intervals (x_i, x_i'), finite or infinite, the sum, $\sum|f(x_i') - f(x_i)|$ is less than M. It is easily shown that functions which satisfy Dirichlet's conditions are of bounded variation. Functions of bounded variation are slightly more general than those which satisfy Dirichlet's conditions in that such a function may have an infinite number of discontinuities and an infinite number of maxima and minima. However, the discontinuities form at most a denumberable set and at every point of discontinuity both $f(x+0)$ and $f(x-0)$ exist. The point to be emphasized is that *there are continuous functions which are not of bounded variation*. If this were not the case du Bois-Reymond's example would contradict the theorem concerning functions of bounded variation.

6. Advances in technique. During these years of the latter half of the nineteenth century there was an intensive search for more powerful weapons of attack. This search culminated in two important discoveries. One was the modern abstract theory of integration, which was first formulated by the French mathematician Henri Lebesgue in 1902 and has come to be known as Lebesgue integration. While this theory was worked out primarily to deal with trigonometric series, it has had a profound effect on the whole field of analysis. Another idea which came out of this period and which has been far-reaching in its affect on analysis is that of the summability of divergent series.

Let $s_1, s_2, \ldots$ be the partial sums of an infinite series $a_1 + a_2 + \ldots$. Let

$$\sigma_n = a_{n1}s_1 + a_{n2}s_2 + \ldots + a_{nn}s_n,$$

where each a_{nk} is a real or complex number. If σ_n tends to a limit as $n \to \infty$ the series $a_1 + a_2 + \ldots$ is said to be summable by means of the set of numbers a_{nk}. If, for example, $a_{nk} = 1/n$, $k = 1, 2, \ldots, n$ then σ_n is the arithmetic mean of the sums $s_1, \ldots, s_n$. If then σ_n tends to a limit the series is said to be summable by arithmetic means or summable $(C, 1)$. The Hungarian mathematician Cesàro was among the first to exploit and extend this general idea of summability. Hence the notation $(C, 1)$.

In 1904 the Hungarian mathematician L. Fejér proved the following theorem **(6, 7)**:

Let the function $f(x)$ be integrable on the interval $[0, 2\pi]$ and let the limits $f(x_0+0)$, $f(x_0-0)$ exist at a point x_0 of this interval. Then the Fourier series

$$\frac{a_0}{2} + \sum_{1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is summable $(C, 1)$ *to* $\{f(x_0 + 0) + f(x_0 - 0)\}/2$.

From this it follows that the Fourier series of a function $f(x)$ *continuous on* $[0, 2\pi]$ *is summable* $(C, 1)$ *to* $f(x)$ *at every point* x.

Fejér's results may be obtained by means which involve only ordinary integration; this is true of all the results we have so far stated concerning trigonometric series. To go further it becomes necessary to use ideas which lead into the Lebesgue theory. It has been noted that du Bois-Reymond gave an example of a continuous function for which the Fourier series diverged at an everywhere dense set E. If a set is everywhere dense on an interval (a, b) then on every interval (a', b') contained in (a, b) there are an infinite number of points of the set. For example the rational numbers are everywhere dense on any interval. There are functions of bounded variation with points of discontinuity forming an everywhere dense set on $[0, 2\pi]$. We are thus in a position which has the appearance of being anomalous.

There are continuous functions for which the Fourier series diverges at an everywhere dense set. These are functions of bounded variation with an everywhere dense set of discontinuities for which the Fourier series converges everywhere.

This brings us to the end of the main events concerning functions integrable in the ordinary sense. The set of divergence of the Fourier series of the continuous function cited above and the set of discontinuities of a function of bounded variation have a common property. Let E be the one or the other of these sets. Let a positive number ϵ, arbitrarily close to zero, be given. There is then a set of non-overlapping open intervals $\alpha_1, \alpha_2, \ldots$ which contain all of E and which is such that the total length of this set of intervals is less than ϵ. Such a set is known as a *null set*. There are null sets which are both non-denumerable and non-dense.

Since the time of du Bois-Reymond there have been other examples of continuous functions whose Fourier series diverge at everywhere dense sets which were null sets, but never at a set other than a null set. Can a continuous function be constructed whose Fourier series diverges at the points of a set which is not a null set? It was a consideration of this and similar problems which led to a study of functions defined on sets other than intervals and eventually to the modern point-set theory and the abstract theories of integration. With this done, there arose another problem.

7. Functions which are not integrable. If $f(x)$ is integrable on $[0, 2\pi]$ in the ordinary sense or in the Lebesgue sense, and if $f(x)$ is represented by a trigonometric series, then this series is the Fourier series of $f(x)$. There are, however, functions $f(x)$ represented by trigonometric series,

$$(8) \qquad f(x) = \frac{c_0}{2} + \sum_1^\infty (c_k \cos kx + d_k \sin kx)$$

which are not integrable, either in the ordinary or Lebesgue sense. An example of such a function is

$$(9) \qquad f(x) = \sum_1^\infty \frac{\sin(n+1)x}{\log(n+1)}.$$

Let T be the class of all such functions. For a given function $f(x)$ in class T, how can the coefficients in (8) be determined?

We now find ourselves at the beginning of a new period with the powerful tools of summability, the point-set theory, and the Lebesgue theory of integration all available and two fundamental questions outstanding:

I′. Is it possible to construct a continuous function whose Fourier series diverges at a set other than a null set?

II′. If $f(x)$ is a function in class T how can the coefficients c_k, d_k in (8) be determined?

I′ is still open. There is some encouragement towards a positive answer in the fact that a function, not continuous but Lebesgue integrable, has been constructed, whose Fourier series diverges everywhere (Kolmogorof's example; **6**, p. 175).

The answer to II′ has come by stages, the final stage in one direction by a Canadian mathematician, R. D. James (**8**, **9**, **10**). We conclude Part I of this study with a general description of the steps that have, only recently, led to the complete answer to II′.

A remarkably penetrating analysis of this problem, and a complete solution, has been given by the French mathematician Arnaud Denjoy (**11**, **12**). Some indication of his methods will be given in Part II of this address. Solutions have also been given by Marcinkiewicz and Zygmund (**13**), by Burkill (**14**, **15**), and by James as noted above.

8. Generalized integrals. The stages on the road to the answer to II′ have paralleled and in a large measure resulted from the search for an answer to a fundamental problem in real variable theory, which we now give in outline (**16**, pp. 140–164).

If we are given the function $f(x)$ which is the derivative of some function $y = F(x)$ then the expectation is that

$$(10) \qquad F(x) - F(a) = \int_a^x f(x)\, dx.$$

In some simple cases, however, this expectation is not realizable. The classic example of this is

$$f(x) = 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, \qquad x \neq 0,$$
$$= 0, \qquad x = 0.$$

The function $f(x)$ is the derivative of

$$F(x) = x^2 \sin \frac{1}{x^2}, \qquad x \neq 0, \qquad F(0) = 0,$$

and is not integrable, either in the ordinary sense or in the Lebesgue sense. Consequently (10) has no meaning. In this case there is only one point, the origin, in the neighbourhood of which $f(x) = F'(x)$ misbehaves, and

$$F(x) - F(0) = \lim_{h \to 0} \int_h^x f(x)\,dx.$$

But there are functions $f(x)$ which are finite derivatives of other functions $F(x)$ and for which the points of misbehaviour are infinite, in fact, more than a null set (**16**, pp. 148–149). In the years 1912–1915 (**17**, **18**), Denjoy devised an operation, now known as Denjoy integration (**16**, p. 158), which is such that if $f(x)$ is the finite derivative of a continuous function $F(x)$, then

$$D \int_a^x f(x)\,dx = F(x) - F(a).$$

This operation involves a denumerably infinite number of stages (**16**, p. 154). Where does this leave us with problem II′?

If the function $f(x)$ in (8) is integrable in the Denjoy sense, special or general, then

$$c_k = \frac{1}{\pi} D \int_{-\pi}^{\pi} f(x) \cos kx\,dx, \qquad d_k = \frac{1}{\pi} D \int_{-\pi}^{\pi} f(x) \sin kx\,dx.$$

This answers II′ for functions in class T which are Denjoy integrable (**5**, p. 488). It does not, however, give the complete answer. There are functions $f(x)$ for which (8) holds which are not integrable even in the Denjoy sense. For example if

$$f(x) = \sum_1^\infty b_n \sin nx, \qquad 0 < x < 2\pi$$

where b_n decreases and tends to zero and $\sum_1^\infty b_n/n$ diverges, then $f(x)$ is not D-integrable (**12**, pp. 42, 43). The function in (9) above is in this category.

During the period that Denjoy was developing his theory of integration, problems in differential equations arose in which the derivatives, $dy/dx = f(x)$, were not integrable. To deal with non-integrable functions, O. Perron considered functions $\psi(x)$, $\phi(x)$ on $[a, b]$ with the following properties. These functions are continuous, $\psi(a) = \phi(a) = 0$ and, if $\underline{D}\,\psi(x)$ is the lower derivative of $\psi(x)$, $\bar{D}\phi(x)$ the upper derivative of $\phi(x)$, then, except possibly for a null set,

$$\text{(i)} \qquad \underline{D}\,\psi(x) \geqslant f(x), \quad \bar{D}\,\phi(x) \leqslant f(x),$$

and for all x except possibly a denumerable set,

$$\text{(ii)} \qquad \underline{D}\,\psi(x) > -\infty, \quad \bar{D}\,\phi(x) < \infty.$$

Let $\Psi(b) = \inf \psi(b)$, $\Phi(b) = \sup \phi(b)$. If $\Psi(b) = \Phi(b)$ the common value is the Perron integral of $f(x)$ over $[a, b]$.

The functions $\psi(x)$ and $\phi(x)$ are called major and minor functions respectively of $f(x)$. It is shown that if the P-integral over $[a, b]$ exists then the P-integral exists over $[a, x]$, $a < x < b$. Consequently, if a function $F(x)$ is continuous on $[a, b]$ and has a finite derivative $f(x)$ at each point of $[a, b]$, then $F(x) - F(a)$ is both a major function and a minor function to $f(x)$. It then follows that

$$F(x) - F(a) = P \int_a^x f(x)\,dx.$$

This method of defining an integral is simpler than that of Denjoy which involves an infinite number of steps. It can be said, however, that Denjoy's method provides a constructive process which can be carried out on $f(x)$ to arrive at the value of the integral. There is no indication in Perron's definition how to determine for a given $f(x)$ the major and minor functions $\psi(x)$, $\phi(x)$. Consequently it can be questioned if Perron has actually defined an integral.

For a long time there was speculation as to the relative generality of these two definitions. Finally, after ten years, it was shown that they are equivalent (**18**, pp. 250, 251; **5**, p. 284). Consequently the Perron integral carries us no further along the road to the answer to II′ than does the integral of Denjoy. Nevertheless, it is Perron's ideas that have been generalized by Marcinkiewicz and Zygmund, by Burkill and by James to give in brief form the complete answer.

9. The work of Marcinkiewicz and Zygmund Burkill and James.

Let

$$\text{(11)} \qquad f(x) = \frac{c_0}{2} + \sum_{1}^{\infty} (c_k \cos kx + d_k \sin kx), \qquad \text{on } -\pi \leqslant x < \pi,$$

where c_k and d_k tend to zero, and let $f(x)$ be defined by periodicity with period 2π outside this interval. Consider the series obtained by integrating formally the right-side term by term,

$$C + \frac{c_0 x}{2} - \sum_1^\infty \frac{d_k \cos kx - c_k \sin kx}{k}. \tag{12}$$

Since the coefficients c_k/k, d_k/k in this series tend to zero more rapidly than the coefficients c_k, d_k in (11), and since (11) converges, it is reasonable to hope that (12) converges. That this is not the case is an illustration of the unpredictable behaviour of trigonometric series. However, it is known that (12) converges, except possibly for a null set. If in (12) $C = 0$ then

$$F(x) = \frac{c_0 x}{2} - \sum_1^\infty \frac{d_k \cos kx - c_k \sin kx}{k} \tag{13}$$

is defined on $-\pi \leqslant x < \pi$ except for at most a null set. Marcinkiewicz and Zygmund (**13**, pp. 35–43), and also Burkill (**14, 15**), define generalized derived numbers and derivatives in such a way that the generalized derivative of $F(x)$ is, except for a null set, the function $f(x)$ in (11). Then in Perron's definition of major and minor functions they replace continuity by a type of mean continuity, and upper and lower derived numbers by these generalized derived numbers and arrive at a generalized integral for which the following is true.

If $f(x)$ is a function in class T then the trigonometric series which represents it, as in (11), is such that

$$c_k = \frac{1}{\pi} \int_\omega f(t) \cos kt\, dt, \qquad d_k = \frac{1}{\pi} \int_\omega f(t) \sin kt\, dt$$

where ω is a set on $(-\pi, \pi)$ which includes all the points of this interval, except a null set.

We come finally to the approach used by James. If the series (11) is formally integrated term by term twice, the resulting series converges everywhere to a continuous function

$$F(x) = \frac{c_0 x^2}{4} - \sum_1^\infty \frac{c_k \cos kx + d_k \sin kx}{k^2} + Cx + C'. \tag{14}$$

The generalized second derivative, $D^2F(x)$, of a function $F(x)$ is the limit as $h \to 0$, h positive, of the ratio

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2}.$$

The upper and lower limits of this ratio are denoted respectively by $\bar{D}^2F(x)$, $\underline{D}^2F(x)$. If $D^2F(x)$ is finite for a point x_0 it follows that the ratio $\{F(x_0+h) + F(x_0-h) - 2F(x_0)\}/h \to 0$ as $h \to 0$, in which case $F(x)$

is said to be smooth at x_0. It is known (**6**, p. 271; **7**, p. 429) that if $F(x)$ is the function defined by (14) then $D^2F(x)$ exists and is equal to the function $f(x)$ defined in (11). Relative to any function $f(x)$ James replaces the Perron conditions by the following.

The functions $M(x)$, $m(x)$ are respectively major and minor functions to $f(x)$ if they are continuous, if $M(a) = M(b) = m(a) = m(b) = 0$, and if $\underline{D}^2M(x) \geqslant f(x)$, $\bar{D}^2m(x) \leqslant f(x)$, except possibly for a null set, $\underline{D}^2M(x) > -\infty$, $\bar{D}^2M(x) < \infty$, except possibly for a denumerable set at the points of which $M(x)$ and $m(x)$ are smooth.

James then defines what he calls the P^2-integral of $f(x)$ as the common bound, if it exists, of the sets of functions $M(x)$, $m(x)$. Actually, if $f(x)$ is the function defined by (11) then this common bound is the function $F(x)$ defined by (14), if the constants C and C' are properly chosen. In general if $f(x)$ is any function in class T then the coefficients in (11) are given by

$$c_k = -\frac{1}{\pi^2} P^2 \int_{-2\pi,0,2\pi} f(t) \cos kt\, dt, \quad d_k = -\frac{1}{\pi^2} P^2 \int_{-2\pi,0,2\pi} f(t) \sin kt\, dt.$$

The significance of the range of integration $(-2\pi, 0, 2\pi)$ will be considered in Part II, §14.

The main points in the development of trigonometric series have now been covered. Their apparent simplicity gives cause for amazement that it has taken so long to settle them. Another cause for amazement is the vast accumulation of results that have arisen as side issues to these main considerations. Some indication of the scope of the whole subject of trigonometric series may be obtained from (**6**).

PART II

Introduction. In this part of the address we give the proofs of the results outlined in Part I. We realize that Theorems 1–7 are in the more elementary parts of any treatise on Fourier series (**6**, pp. 14–63; **7**, pp. 399–416). We give them in order to make our story complete, and also to emphasize the ease and brevity with which the methods of modern mathematics handle problems which for so long baffled the ablest mathematicians. Except for Theorem 3, which can be omitted, no knowledge beyond that which includes ordinary integration is presumed before Section 13. Then a knowledge of the elements of the Lebesgue theory (**7**, Chapters X–XII; **16**, Chapters II–V) is required.