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CRM Short Thematic Program
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and Entire Holomorphic Curves
and Algebraic Varieties
June 3–28, 2013
Centre de Recherches Mathématiques,
Université de Montréal, Québec, Canada

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Rational Points, Rational Curves, and Entire Holomorphic Curves on Projective Varieties

Preface

Diophantine geometry and the study of rational points on algebraic varieties have greatly influenced and continue to revolutionize modern algebraic geometry. The general philosophy is that "the geometry determines arithmetic behaviour".

It is conjectured that there are many rational points (at least after a finite extension of the base field) on "special" varieties, a class of varieties introduced by Campana which includes rationally connected and Calabi–Yau varieties. Conversely, the conjecture of Bombieri–Lang predicts that varieties of general type should have "few" rational points. These conjectures and philosophy are the subject of intense activity.

One of the fascinating aspects of these questions is their relations with complex analytic geometry. After Lang and Vojta, we expect that arithmetic properties of an algebraic variety correspond via value distribution theory to complex hyperbolic properties through a dictionary which translates properties of rational points into properties of holomorphic curves. For instance, for special varieties (in the sense of Campana) it is conjectured that the Kobayashi pseudo-distance is trivial, and that such varieties have many holomorphic curves, and in the simply connected case, many rational curves. Much work has been done to establish the expected properties which are the complex geometric counterpart to the above mentioned conjectured results in arithmetic.

In June 2013 a thematic month around these topics was organized at the CRM in Montreal supported in part by an ANR project grant. It was also generously supported by the NSF and locally supported by the CRM and CIRGET. Specialists from around the globe introduced the latest advances on the subject and specialized mini-courses were given geared to young researchers.

In this proceedings volume we gather the lecture notes of some of the mini-courses of the thematic month and contributed papers by key specialists in these areas.

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Some applications of p-adic uniformization to algebraic dynamics

Ekaterina Amerik

ABSTRACT. We describe how certain simple p-adic techniques can be applied to get information about iterated orbits of algebraic points under a rational self-map of an algebraic variety defined over a number field.

The purpose of these notes is to give a brief survey of several topics at the limit of geometry and arithmetics, where some fairly elementary p-adic methods have led to highly non-trivial results. These results are recent but not brand-new: all proofs have been published elsewhere. My hope and reason for putting them together is that this might facilitate further progress, in particular by young mathematicians or those who are new to the field. Since I am a geometer which only meets arithmetics by accident, the point of view in these notes is quite biased. The reader is encouraged to consult the texts by other people who have contributed to the subject: in particular, the forthcoming book by Bell, Ghioca and Tucker [**BGTbook**] promises to be very interesting.

The notes are written for the proceedings volume of CRM Montreal thematic program "Rational points, rational curves and entire holomorphic curves on algebraic varieties" in June 2013. During the writing of the notes, I was also preparing a mini-course on the subject for ANR BirPol and Fondation Del Duca meeting "Groupes de transformations" in Rennes in June 2014. I am grateful to the organizers of both activities for giving me this opportunity to speak. Thanks also to Dragos Ghioca for sending me a preliminary version of [**BGTbook**] and answering a few questions.

1. A motivation: potential density

Let X be a projective variety over a field K.

DEFINITION 1. *Rational points of X are potentially dense over K (or, as one also sometimes says, X is potentially dense over K) if there is a finite extension L of K such that the L-points are Zariski dense in X.*

The reason for looking at the potential density rather than at the density of K-rational points is that the potential density behaves much better from the geometric point of view. Indeed, even a plane conic over the rationals can have a dense set of rational points or no rational points at all; whereas if we look at the potential density, we may at least hope that the varieties which share similar geometric properties should be potentially dense (or not potentially dense) all at once.

If X is *rational* (that is, birational to $\mathbb{P}^n$), or, more generally, *unirational* (that is, dominated by $\mathbb{P}^n$) over $\bar{K}$, then rational points are obviously potentially dense on X. Indeed, choose L such that the unirationality map $f : \mathbb{P}^n \dashrightarrow X$ is defined over L. Then the images of L-points of $\mathbb{P}^n$ are L-points of X and they are Zariski dense in X since f is dominant and L-points are dense on $\mathbb{P}^n$. More generally, a variety dominated by a potentially dense variety is itself potentially dense.

It is certainly not true in general that a variety which dominates a potentially dense variety is itself potentially dense. However this is true in an important particular case: if $f : X \to Y$ is a finite étale morphism and Y is potentially dense, then so is X. This follows from Chevalley-Weil theorem (see for example [**S**]). The idea is that points in the inverse image of $x \in X(K)$ are points over finite extensions of degree equal to $deg(f)$ and ramified only at a fixed (that is, independent of x) finite set of places. There is only a finite number of such extensions.

Unirational varieties share many other properties of the projective space. For instance, the tensor powers of the canonical line bundle K_X on such a variety X have no sections: indeed, a section of $K_X^{\otimes m}$, $m > 0$, would pull back to $\mathbb{P}^n$ and give a section of a tensor power of $K_{\mathbb{P}^n}$ (by Hartogs' extension theorem), but no such section exists.

On the opposite geometric end, we have the *varieties of general type*: these are the varieties on which the tensor powers $K_X^{\otimes m}$ have "lots of sections". More precisely, a smooth projective variety X is said to be *of general type* if the map defined by the linear system $|K_X^{\otimes m}|$ is birational to its image for some $m > 0$. The simplest examples are curves of genus $g \geq 2$, or smooth hypersurfaces of degree $\geq n+2$ in $\mathbb{P}^n$. The following conjecture is very famous:

CONJECTURE 2. *(Lang-Vojta) A smooth projective variety X which is of general type cannot be potentially dense over a number field K.*

Up to now this is known only for curves and for subvarieties of abelian varieties, by the work of G. Faltings.

Lang-Vojta conjecture implies that varieties dominating a variety of general type cannot be potentially dense over a number field. One might ask whether this should lead to a geometric characterization of potentially dense varieties.

The naive guess is wrong: one can construct a surface which is not of general type and does not dominate any curve of genus $g \geq 2$, yet it is not potentially dense, since it admits a finite étale covering which *does* map onto a curve of higher genus, and the potential density is stable under finite étale coverings. This seems to be first observed by Colliot-Thélène, Skorobogatov and Swinnerton-Dyer in [**CSS**]. The idea is to take an elliptic surface X over $\mathbb{P}^1$ with suitably many double fibers, so that locally the map to $\mathbb{P}^1$ looks like $(x, y) \mapsto u = x^2$. Then the ramified covering C of the base which eliminates these multiple fibers (that is, locally looks like $z \mapsto u = z^2$, so that the fibered product is singular and its normalization is étale over X) will have genus at least two.

F. Campana suggests in [**C**] that the potentially dense varieties are exactly the so-called *special varieties*. Roughly speaking, these are the varieties which do not dominate *orbifolds of general type*: if $f : X \to B$ is a fibration with certain good properties (which are achieved on suitable birational models), one can define an orbifold canonical bundle $K_B + \Delta$ by taking into account the multiple fibers of f, and this bundle should not have too many section. For the moment, proving this, or even the "easier" direction that special varieties should be potentially dense, looks quite out of reach.

In any case, all existing philosophy seems to imply that the varieties with negative canonical bundle (the *Fano varieties*) or trivial canonical bundle must be potentially dense. There is a reasonable amount of evidence for this in the Fano case: indeed many Fano varieties are known to be unirational, and when the unirationality is unknown the potential density still can sometimes be proved (see for example [**HarT**]). Also, potential density is known for tori (and it shall be explained in this survey in a particularly elementary way). But the case of simply-connected varieties with trivial canonical class remains mysterious: indeed, even for a general K3 surface the answer is unknown, and moreover there is no example of a potentially dense K3 surface with cyclic Picard group (that is, "general" in the moduli of polarized K3).

Bogomolov and Tschinkel [**BT**] proved the potential density of elliptic K3 surfaces.

THEOREM 3. *Let X be a K3 surface over a number field K. If X admits an elliptic fibration, then X is potentially dense.*

Idea of proof: Construct a multisection C which is a rational curve (so has a lot of rational points) and is non-torsion, that is, the difference of at least two of its points on a general fiber is non-torsion in the jacobian of this fiber. Then one can move C along the fibers by "fiberwise multiplying it by an integer" and produce many new rational points in this way.

More generally, let X be a variety equipped with a rational self-map $f : X \dashrightarrow X$, both defined over a number field K (such as the fiberwise multiplication by k on an elliptic surface; this exists for any $k \in \mathbb{Z}$ when the surface has a section and for suitable k if not). It is a natural idea to use f to produce many rational points on X: indeed f sends rational points to rational points.

This approach has first been worked out by Claire Voisin and myself [**AV**] to give the first example of a simply-connected variety with trivial canonical class which has Picard number one (so is "general" in the polarized moduli space) and has potentially dense rational points. Our example is as follows.

Let V be a cubic in $\mathbb{P}^5$ and $X = \mathcal{F}(V) \subset Gr(1,5)$ be the variety parameterizing the lines on V. A simple computation shows that X is a smooth simply-connected fourfold with trivial canonical bundle. Moreover it can be seen as a higher dimensional analogue of a K3 surface: as shown by Beauville and Donagi [**BD**], X is an irreducible holomorphic symplectic manifold (that is, $H^{2,0}(X)$ is generated by a single nowhere degenerate form σ), deformation equivalent to the second punctual Hilbert scheme $Hilb^2(S)$, where S is a K3 surface (and actually isomorphic to $Hilb^2(S)$ when the cubic V is pfaffian).

PROPOSITION 4. *(C. Voisin) X admits a dominant rational self-map $f : X \dashrightarrow X$ of degree 16.*

Sketch of proof: Let us describe the construction: for a general line l on V, there is a unique plane P tangent to V along l (indeed the normal bundle $N_{l,V} = \mathcal{O}_l \oplus \mathcal{O}_l \oplus \mathcal{O}_l(1)$, which makes this unicity appear on the infinitesimal level). One defines $f(l)$ as the only line which is residual to l in the intersection $P \cap V$, and one shows (using, for example, Mumford's trick on algebraic cycles and differential forms) that f multiplies σ by -2.

THEOREM 5. *([**AV**]) For "most" cubic 4-folds V defined over a number field, the corresponding variety $X = \mathcal{F}(V)$ (which is defined over the same number field) has cyclic geometric Picard group and is potentially dense.*

What is meant by "most" can be made precise, but this is a rather complicated condition. Since it is not related to our main subject, let us only mention that the parameter point of the cubic fourfold in question should be outside of a certain thin subset, like in Hilbert irreducibility theorem.

The proof, too, is long and involved; in fact most of my contribution to the main subject of these notes grew out of a search for a more elementary argument. Let us only mention the starting point: we consider a family of birationally abelian surfaces $\Sigma_t, t \in T$ covering X (the existence of such a family was observed by Claire Voisin in relation to Kobayashi pseudometric issues) and remark that since rational points are potentially dense on Σ_t for algebraic t, it is enough to find an algebraic t such that the iterates $f^k(\Sigma_t)$ are Zariski dense in X.

It turns out to be surprisingly difficult to show by the methods of complex geometry that the iterates of something algebraic are Zariski dense. Let me illustrate this point by explaining the difference with the transcendental situation.

The following theorem has been proved by Campana and myself in the complex geometry setting.

THEOREM 6. *Let X be a projective variety and $f : X \dashrightarrow X$ a dominant rational self-map, both defined over an algebraically closed field K. Then there is a dominant rational map $g : X \dashrightarrow T$ to a projective variety T, such that $gf = g$ and for a sufficiently general point $x \in X$, the fiber of f through x is the Zariski closure of the iterated orbit $O_f(x) = \{f^k(x), k \in Z\}$.*

One can always Stein-decompose g to arrive to a map with connected fibers preserved by a power of f. The theorem thus implies that if no power of f preserve a non-trivial rational fibration (and this is something which often can be easily established by geometric methods, see for example [**AC**], theorem 2.1 and corollary 2.2), the orbit of a sufficiently general point is Zariski dense. If, on the contrary, some power of f does preserve a fibration, then this is obviously not the case.

Unfortunately "sufficiently general" in the theorem means "outside a countable union of proper subvarieties" (the theorem is proved by looking at the Chow components parameterizing f-invariant subvarieties and discarding the families which do not dominate X). That is, when the field K is uncountable, most $x \in X$ are indeed general in this sense; but the theorem does not give any information when K is countable, since it might happen that no K-point is sufficiently general!

In particular, we still do not know whether there are algebraic points on the variety of lines of a cubic fourfold which have Zariski-dense iterated orbit under f. What we do know is that f does not preserve a rational fibration, and neither do its powers, by [**AC**], theorem 2.1; but apriori the iterated orbits of algebraic points can have smaller Zariski-closure than those of general complex points.

One would like to conjecture that in reality it never happens: this is already implicit in [**AC**].

CONJECTURE 7. *Let X be an algebraic variety with a dominant rational self-map $f : X \dashrightarrow X$ defined over a number field K. Consider the map $g : X \dashrightarrow T$*

from theorem 6, and let d denote its relative dimension. Then there exists an algebraic point $x \in X(\bar{\mathbb{Q}})$ such that the dimension of the Zariski closure of $O_f(x)$ is equal to d.

Some less general versions have been formulated by other authors; for instance, the following conjecture has been made by Shouwu Zhang. For X a smooth projective variety, let us call an endomorphism $f : X \to X$ *polarized*, if there is an ample line bundle L on X such that $f^*(L) = L^{\otimes q}$ with $q > 1$.

CONJECTURE 8. *(Zhang) Let X be a smooth projective variety and $f : X \to X$ be a polarized endomorphism of X defined over a number field K. Then there exists a point $x \in X(\bar{\mathbb{Q}})$ with Zariski-dense iterated orbit $O_f(x)$.*

Note that a polarized endomorphism cannot preserve a fibration. Indeed, otherwise let F be a fiber; one then should have $deg(f|_F) = deg(f)$. But the former is $q^{dim(F)}$ and the latter $q^{dim(X)}$, a contradiction.

Therefore Zhang's conjecture would follow from conjecture 7. Indeed, since no power of f preserve fibrations, T is a point and if the conjecture 7 is true, there is an algebraic point with Zariski-dense orbit.

In what follows, we shall try to explain some p-adic ideas towards the proof of this conjecture.

One should mention, though, that there is no hope to prove the potential density of all special varieties using rational self-maps, as the self-maps do not always exist. For instance, Xi Chen [**Ch**] proved that a general K3 surface does not admit a non-trivial rational self-map. Nevertheless, an interesting example (variety of lines of a cubic fourfold) has been studied in this way, and hopefully more shall follow.

Independently of potential density issues, conjecture 7, as well as its weaker versions, looks quite hard. When the ambient variety has a large family of rational self-maps, for instance, is rational, Proposition 13 below indicates that it should be true for a "sufficiently general" of them, in some sense. For self-maps of a particular shape, one can perform explicit computations. Xie Junyi [**X**], building on results by myself [**A**] and by Serge Cantat [**Can**], has remarked that the conjecture holds for birational maps of surfaces.

Answering a question by one of the referees, let us also mention that Bell, Ghioca and Reichstein have recently remarked that there is an analogue of theorem 6 for a semigroup of rational self-maps rather then a single map; therefore it makes sense to make an analogous conjecture for e.g. finitely generated semigroups.

2. Near a fixed point

While working on problems of holomorphic dynamics, one is often led to consider the behaviour of the map in a neighbourhood of a fixed point. In [**ABR**], we have tried to work out some rudiments of a similar approach in algebraic geometry in order to simplify and render more explicit the proof of potential density of the variety of lines on a cubic fourfold from [**AV**]. Somewhat later, we have learned that similar ideas were exploited by Ghioca and Tucker in order to settle a case of the so-called *dynamical Mordell-Lang conjecture* to which we shall return in the next section.

Endomorphisms often have periodic points: for instance, a theorem by Fakhruddin [**F**] asserts that a polarized endomorphism has a Zariski-dense subset of periodic points. Replacing f by a power if necessary, we may assume that some periodic point is actually fixed.

If $X = \mathcal{F}(V)$ is the variety of lines of a cubic in $\mathbb{P}^5$ and $f : X \dashrightarrow X$ is the rational map which sends a general line l to the line l' which is residual to l in the intersection of V with the plane tangent to V along l, then the fixed points are, obviously, the lines such that there is a plane tritangent to V along this line (and not contained in the indeterminacy locus, that is, this tritangent plane should be the only plane tangent to V along l). An explicit computation shows that such lines form a surface on X, and no component of this surface is contained in the indeterminacy locus (one can, for instance, remark that the fixed surface is certainly lagrangian, because of the identity $f^*\sigma = -2\sigma$, where σ is the symplectic form, and that the indeterminacy locus is not lagrangian because of the computations in [**A0**]; but there is probably a much easier way).

2.1. Linearization in a p-adic neighbourhood. Let X be arbitrary, and let q be a fixed point of a rational map $f : X \dashrightarrow X$. Assume that everything is defined over a number field K. We shall denote by $\mathcal{O}_K$ the ring of integers, by $\mathfrak{p} \subset \mathcal{O}_K$ an ideal, by $O_\mathfrak{p}$ and $K_\mathfrak{p}$ the $\mathfrak{p}$-adic completions.

Our starting observation is that for a suitable $\mathfrak{p}$, one can find a $\mathfrak{p}$-adic neighbourhood $O_{\mathfrak{p},q}$ (that is, the set of $\mathfrak{p}$-adic points reducing to the same point as q modulo $\mathfrak{p}$ in a suitable model of X) which is invariant under f, and f is well-defined there.

One can define and describe $O_{\mathfrak{p},q}$ in a very down-to-earth way, without talking about models, by a $\mathfrak{p}$-adic version of the implicit function theorem.

Namely, following [**ABR**], choose an affine neighbourhood $U \subset X$ of q, such that the restriction of f to U is regular. By Noether normalisation lemma, there is a finite K-morphism $\pi = (x_1, \dots, x_n) : U \longrightarrow \mathbb{A}^n_K$ to the affine space, which is étale at q and which maps q, say, to 0. Then the K-algebra $\mathcal{O}(U)$ is integral over $K[x_1, \dots, x_n]$, i.e., it is generated over $K[x_1, \dots, x_n]$ by some regular functions $x_{n+1}, \dots, x_m$ integral over $K[x_1, \dots, x_n]$. We can view $x_{n+1}, \dots, x_m$ and $f^*x_1, \dots, f^*x_m$ as power series in $x_1, \dots, x_n$ with coefficients in K (indeed the coordinate ring of U is embedded into the local ring of q and the latter is embedded into its completion). Since everything is algebraic over $K(x_1, \dots, x_n)$, one can show that all coefficients lie in a finitely generated $\mathbb{Z}$-algebra (this goes back to Eisenstein for $n = 1$, see [**ABR**], lemma 2.1). In particular, for almost all primes $\mathfrak{p}$, the coefficients of our power series are $\mathfrak{p}$-integral. Take such a $\mathfrak{p}$ satisfying the following extra condition: for $n < i \leq m$, let P_i be the minimal polynomial of x_i over $x_1, \dots, x_n$. We want $x_i(q)$ to be a simple root of $P_i(q)$ modulo $\mathfrak{p}$ (this condition is obviously expressed in terms of the non-vanishing of derivatives modulo $\mathfrak{p}$, and thus also holds for almost all $\mathfrak{p}$).

Set

$$O_{\mathfrak{p},q,s} = \{t \in U(K_\mathfrak{p}) | x_i(t) \equiv x_i(q) \pmod{\mathfrak{p}^s} \ \textit{for} \ 1 \leq i \leq m\},$$

and let $O_{\mathfrak{p},q} = O_{\mathfrak{p},q,1}$.

View all our functions x_i, f^*x_i as elements of $\mathcal{O}_\mathfrak{p}[[x_1, \dots, x_n]]$. The following properties are then obvious by construction.

PROPOSITION 9. *([**ABR**], Prop. 2.2) (1) The functions $x_1, \dots x_n$ give a bijection between $O_{\mathfrak{p},q,s}$ and the n-th cartesian power of $\mathfrak{p}^s$.*

(2) The set $O_{\mathfrak{p},q}$ contains no indeterminacy points of f.