

# Graduate Texts in Mathematics

**Ronald G. Douglas**

## **Banach Algebra Techniques in Operator Theory**

**Second Edition**

**巴拿赫代数在算子理论  
中的应用 第2版**

Springer-Verlag

世界图书出版公司

Ronald G. Douglas

# Banach Algebra Techniques in Operator Theory

Second Edition



Springer

书 名: Banach Algebra Techniques in Operator Theory 2nd ed.  
作 者: R. G. Douglas  
中译名: 巴拿赫代数在算子理论中的应用 第2版  
出 版 者: 世界图书出版公司北京公司  
印 刷 者: 北京世图印刷厂  
发 行: 世界图书出版公司北京公司 (北京朝内大街 137 号 100010)  
联系电话: 010-64015659, 64038347  
电子信箱: kjsk@vip.sina.com  
开 本: 24 印 张: 8.5  
出版年代: 2003 年 6 月  
书 号: 7-5062-5962-1 / O · 381  
版权登记: 图字:01-2003-3773  
定 价: 28.00 元

世界图书出版公司北京公司已获得 Springer-Verlag 授权在中国大陆  
独家重印发行。

Ronald G. Douglas  
Department of Mathematics  
Texas A & M University  
College Station, TX 77843-1248  
USA

*Editorial Board*

S. Axler  
Mathematics Department  
San Francisco State  
University  
San Francisco, CA 94132  
USA

F.W. Gehring  
Mathematics Department  
University of Michigan  
Ann Arbor, MI 48109  
USA

K.A. Ribet  
Department of Mathematics  
East Hall University of  
California at Berkeley  
Berkeley, CA 94720-3840  
USA

---

Mathematics Subject Classification (1991): 46Bxx, 46Jxx, 46Lxx, 47Bxx

---

Library of Congress Cataloging-in-Publication Data  
Douglas, Ronald G.

Banach algebra techniques in operator theory / Ronald G. Douglas. — 2nd ed.  
p. cm. — (Graduate texts in mathematics ; 179)

Includes bibliographical references (p. — ) and index.

ISBN 0-387-98377-5 (hardcover : alk. paper)

1. Operator theory. 2. Banach algebras. 3. Hilbert space

I. Title. II. Series.

QA329.D68 1998

515'.724—dc21

97-42889

Printed on acid-free paper.

The first edition of this book was published by Academic Press, New York, © 1972.

© 1998 Springer-Verlag New York, Inc.

All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer-Verlag New York, Inc., 175 Fifth Avenue, New York, NY 10010, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.

The use of general descriptive names, trade names, trademarks, etc., in this publication, even if the former are not especially identified, is not to be taken as a sign that such names, as understood by the Trade Marks and Merchandise Marks Act, may accordingly be used freely by anyone.

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom.  
Reprinted in China by Beijing World Publishing Corporation, 2003

9 8 7 6 5 4 3 2 1

ISBN 0-387-98377-5 Springer-Verlag New York Berlin Heidelberg SPIN 10656861

## Preface to the Second Edition

In the quarter century since the first edition of this book appeared, tremendous development has occurred in operator theory and the topics covered here. However, the new edition remains unchanged except that several mistakes and typographical errors have been corrected. Further, a brief report on the current state of the double-asterisk, open, problems is given along with references. No attempt is made to describe other progress that has been made in the study of Toeplitz operators and related topics nor has the bibliography been updated.

Still, it is hoped that a new generation of students will find useful the introduction to operator theory given here.

College Station, Texas  
July 1997

Ronald G. Douglas

## Preface to the First Edition

Operator theory is a diverse area of mathematics which derives its impetus and motivation from several sources. It began as did practically all of modern analysis with the study of integral equations at the end of the last century. It now includes the study of operators and collections of operators arising in various branches of physics and mechanics as well as other parts of mathematics and indeed is sufficiently well developed to have a logic of its own. The appearance of several monographs on recent studies in operator theory testifies both to its vigor and breadth.

The intention of this book is to discuss certain advanced topics in operator theory and to provide the necessary background for them assuming only the standard senior-first year graduate courses in general topology, measure theory, and algebra. There is no attempt at completeness and many "elementary" topics are either omitted or mentioned only in the problems. The intention is rather to obtain the main results as quickly as possible.

The book begins with a chapter presenting the basic results in the theory of Banach spaces along with many relevant examples. The second chapter concerns the elementary theory of commutative Banach algebras since these techniques are essential for the approach to operator theory presented in the later chapters. Then after a short chapter on the geometry of Hilbert space, the study of operator theory begins in earnest. In the fourth chapter operators on Hilbert space are studied and a rather sophisticated version of the spectral theorem is obtained. The notion of a  $C^*$ -algebra is introduced and used throughout the last half of this chapter. The study of compact operators and Fredholm operators is taken up in the fifth chapter along with certain ancillary results concerning ideals in  $C^*$ -algebras. The approach here is a bit unorthodox but is suggested by modern developments.

The last two chapters are of a slightly different character and present a systematic development including recent research of the theory of Toeplitz operators. This

latter class of operators has attracted the attention of several mathematicians recently and occurs in several rather diverse contexts.

In the sixth chapter certain topics from the theory of Hardy spaces are developed. The selection is dictated by needs of the last chapter and proofs are based on the techniques obtained earlier in the book. The study of Toeplitz operators is taken up in the seventh chapter. Most of what is known in the scalar case is presented including Widom's result on the connectedness of the spectrum.

At the end of each chapter there are source notes which suggest additional reading along with giving some comments on who proved what and when. Although a reasonable attempt has been made in the latter chapters at citing the appropriate source for important results, omissions have undoubtedly occurred. Moreover, the absence of a reference should not be construed to mean the result is due to the author.

In addition, following each chapter is a large number of problems of varying difficulty. The purposes of these are many: to allow the reader to test his understanding; to indicate certain extensions of the theory which are now accessible; to alert the reader to certain important and related results of which he should be aware along with a hint or a reference for the proof; and to point out certain questions for which the answer is not known. These latter questions are indicated by a double asterisk; a single asterisk indicates a difficult problem.

Stony Brook, New York  
August 1971

Ronald G. Douglas

# Acknowledgments

This book began as a set of lecture notes for a course given at the University of Michigan in Spring, 1968 and again at SUNY at Stony Brook in the academic year, 1969–1970.

I am indebted to many people in the writing of this book. Most of all I would like to thank Bruce Abrahamse who prepared the original notes and who has been a constant source of suggestions and constructive criticism writing portions of later versions. In addition, I would like to thank many friends and colleagues for their many suggestions and, in particular, Paul Halmos, Carl Percy, Pasquale Porcelli, Donald Sarason, and Allen Shields with whom I have learned many of the things presented in this book. Special thanks are due to Berrien Moore III who read and criticized the entire manuscript and to Joyce Lemen, Dorothy Lentz, and Carole Alberghine for typing the various versions of this manuscript. Lastly, I would like to thank the National Science Foundation and the Alfred E. Sloan Foundation for various support during the writing of this book.

Stony Brook, New York  
August 1971

Ronald G. Douglas

MAY 52/08



# Symbols and Notation

$A$	47	$H_0^p$	144
$\text{Aut}( )$	55	$H^\infty$	25
$\mathbb{C}$	1	$H_0^\infty$	26
$\mathbb{C}^n$	61	$H^\infty + C(\mathbb{T})$	145
$C(X)$	1	$I_E$	17
$\chi_n$	25	$i_i$	166
$c_0(\mathbb{Z}^+)$	6	$j$	115
$\mathfrak{S}_T$	85	$\ker$	75
$\mathbb{D}$	48	$l^1(\mathbb{Z}^+)$	6
$\partial F_\lambda$	177	$l^1(\mathbb{Z})$	50
$\exp$	34	$l^2(\mathbb{Z}^+)$	62
$\mathfrak{F}( )$	113	$l^2(\mathbb{Z})$	82
$\mathfrak{F}_n$	115	$l^\infty(\mathbb{Z}^+)$	6
$F_\lambda$	177	$L^1$	23
$\mathcal{G}$	33	$L^2$	63
$\Gamma \mathfrak{B}$	37	$L^p$	23
$h$	163	$L^\infty$	23
$H^1$	25	$\mathfrak{L}( )$	20
$H_0^1$	144	$\mathfrak{L}\mathfrak{C}( )$	108
$H^2$	64	$\mathfrak{L}\mathfrak{F}( )$	108
$H^p$	133	$\wedge \mathfrak{B}$	33
		$M\mathfrak{B}$	36

## xvi Symbols and Notation

$M_\varphi$	81	$T_\varphi$	158
$M_\infty$	138	$T_K$	112
$M(X)$	19	$\mathfrak{T}(\ )$	160
$P$	158	$U$	82
$\mathcal{P}$	47	$U_+$	87
$\mathcal{P}_+$	47	$U_+^{(n)}$	121
$PC$	157	$w(\ )$	104
$P\mathcal{M}$	78	$W(\ )$	104
$QC$	157	$\mathfrak{B}_T$	92
$\mathbb{R}$	3	$\mathbb{Z}$	6
$r(\ )$	38	$\  \ \ _\infty$	1
$\mathcal{R}(\ )$	52	$*$	5, 75, 107
$\text{ran}$	75	$(\ )_1$	9
$\rho(\ )$	38	$\  \ \ _p$	23
$\rho_c(\ )$	171	$\oplus$	27, 72
$\sigma(\ )$	38	$\perp$	28, 65
$\Sigma$	3	$\otimes$	29, 56, 73, 106
$\mathbb{T}$	25	$(, )$	58
		$'$	98

# Contents

Preface to the Second Edition	v
Preface to the First Edition	vii
Acknowledgments	ix
Symbols and Notation	xv
<b>1 Banach Spaces</b>	<b>1</b>
1 The Banach Space of Continuous Functions	2
2 Abstract Banach Spaces	2
3 The Conjugate Space of Continuous Linear Functionals	5
4 Examples of Banach spaces: $c_0$ , $l^1$ , and $l^\infty$	6
5 Weak Topologies on Banach Spaces	8
6 The Alaoglu Theorem	9
7 The Hahn–Banach Theorem	10
8 The Conjugate Space of $C([0, 1])$	12
9 The Open Mapping Theorem	21
10 The Lebesgue Spaces: $L^1$ and $L^\infty$	23
11 The Hardy Spaces: $H^1$ and $H^\infty$	25
Notes	26
Exercises	26
<b>2 Banach Algebras</b>	<b>30</b>
1 The Banach Algebra of Continuous Functions	30
2 Abstract Banach Algebras	31

3	Abstract Index in a Banach Algebra	33
4	The Space of Multiplicative Linear Functions	36
5	The Gelfand Transform	37
6	The Gelfand–Mazur Theorem	39
7	The Gelfand Theorem for Commutative Banach Algebras	41
8	The Spectral Radius Formula	42
9	The Stone–Weierstrass Theorem	43
10	The Generalized Stone–Weierstrass Theorem	43
11	The Disk Algebra	47
12	The Algebra of Functions with Absolutely Convergent Fourier Series	50
13	The Algebra of Bounded Measurable Functions	52
	Notes	53
	Exercises	53
<b>3</b>	<b>Geometry of Hilbert Space</b>	<b>58</b>
1	Inner Product Spaces	58
2	The Cauchy–Schwarz Inequality	59
3	The Pythagorean Theorem	60
4	Hilbert Spaces	61
5	Examples of Hilbert Spaces: $\mathbb{C}^n$ , $l^2$ , $L^2$ , and $H^2$	61
6	The Riesz Representation Theorem	66
7	The Existence of Orthonormal Bases	69
8	The Dimension of Hilbert Spaces	70
	Notes	71
	Exercises	71
<b>4</b>	<b>Operators on Hilbert Space and <math>C^*</math>-Algebras</b>	<b>74</b>
1	The Adjoint Operator	75
2	Normal and Self-adjoint Operators	77
3	Projections and Subspaces	78
4	Multiplication Operators and Maximal Abelian Algebras	80
5	The Bilateral Shift Operator	82
6	$C^*$ -Algebras	83
7	The Gelfand–Naimark Theorem	84
8	The Spectral Theorem	85
9	The Functional Calculus	85
10	The Square Root of Positive Operators	86
11	The Unilateral Shift Operator	87
12	The Polar Decomposition	88

13	Weak and Strong Operator Topologies .....	91
14	$W^*$ -Algebras .....	92
15	Isomorphisms of $L^\infty$ -Spaces .....	94
16	Normal Operators with Cyclic Vectors .....	95
17	Maximal Abelian $W^*$ -Algebras .....	97
18	$*$ -Homomorphisms of $C^*$ -Algebras .....	100
19	The Extended Functional Calculus .....	101
20	The Fuglede Theorem .....	103
	Notes .....	104
	Exercises .....	104
<b>5</b>	<b>Compact Operators, Fredholm Operators, and Index Theory</b> .....	<b>108</b>
1	The Ideals of Finite Rank and Compact Operators .....	108
2	Approximation of Compact Operators .....	110
3	Examples of Compact Operators: Integral Operators .....	112
4	The Calkin Algebra and Fredholm Operators .....	113
5	Atkinson's Theorem .....	114
6	The Index of Fredholm Operators .....	115
7	The Fredholm Alternative .....	116
8	Volterra Integral Operators .....	118
9	Connectedness of the Unitary Group in a $W^*$ -Algebra .....	119
10	Characterization of Index .....	123
11	Quotient $C^*$ -Algebras .....	124
12	Representations of the $C^*$ -Algebra of Compact Operators .....	126
	Notes .....	129
	Exercises .....	129
<b>6</b>	<b>The Hardy Spaces</b> .....	<b>133</b>
1	The Hardy Spaces: $H^1$ , $H^2$ , and $H^\infty$ .....	133
2	Reducing Subspaces of Unitary Operators .....	135
3	Beurling's Theorem .....	136
4	The F. and M. Riesz Theorem .....	137
5	The Maximal Ideal Space of $H^\infty$ .....	138
6	The Inner–Outer Factorization of Functions in $H^2$ .....	141
7	The Modulus of Outer Functions .....	141
8	The Conjugates of $H^1$ and $L^\infty/H_0^\infty$ .....	144
9	The Closedness of $H^\infty + C$ .....	145
10	Approximation by Quotients of Inner Functions .....	145
11	The Gleason–Whitney Theorem .....	146

12 Subalgebras between $H^\infty$ and $L^\infty$ .....	147
13 Abstract Harmonic Extensions .....	149
14 The Maximal Ideal Space of $H^\infty + C$ .....	150
15 The Invertibility of Functions in $H^\infty + C$ .....	152
Notes .....	153
Exercises .....	153
<b>7 Toeplitz Operators</b> .....	<b>158</b>
1 Toeplitz Operators .....	158
2 The Spectral Inclusion Theorem .....	160
3 The Symbol Map .....	160
4 The Spectrum of Self-adjoint Toeplitz Operators .....	163
5 The Spectrum of Analytic Toeplitz Operators .....	164
6 The $C^*$ -Algebra Generated by the Unilateral Shift .....	164
7 The Invertibility of Toeplitz Operators with Continuous Symbol ..	166
8 The Invertibility of Unimodular Toeplitz Operators and Prediction Theory .....	167
9 The Spectrum of Toeplitz Operators with Symbol in $H^\infty + C$ ....	169
10 The Connectedness of the Essential Spectrum .....	174
11 Localization to the Center of a $C^*$ -Algebra .....	177
12 Locality of Fredholmness for Toeplitz Operators .....	178
Notes .....	178
Exercises .....	181
<b>References</b> .....	<b>185</b>
<b>Index</b> .....	<b>191</b>

## Chapter 1

# Banach Spaces

**1.1** We begin by introducing the most representative example of a Banach space. Let  $X$  be a compact Hausdorff space and let  $C(X)$  denote the set of continuous complex-valued functions on  $X$ . For  $f_1$  and  $f_2$  in  $C(X)$  and  $\lambda$  a complex number, we define:

- (1)  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ ;
- (2)  $(\lambda f_1)(x) = \lambda f_1(x)$ ; and
- (3)  $(f_1 f_2)(x) = f_1(x) f_2(x)$ .

With these operations  $C(X)$  is a commutative algebra with identity over the complex field  $\mathbb{C}$ .

Each function  $f$  in  $C(X)$  is bounded, since it follows from the fact that  $f$  is continuous and  $X$  is compact that the range of  $f$  is a compact subset of  $\mathbb{C}$ . Thus the least upper bound of  $|f|$  is finite; we call this number the norm of  $f$  and denote it by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

The following properties of the norm are easily verified:

- (1)  $\|f\|_\infty = 0$  if and only if  $f = 0$ ;
- (2)  $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$ ;
- (3)  $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$ ; and
- (4)  $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ .

We define a metric  $\rho$  on  $C(X)$  by  $\rho(f, g) = \|f - g\|_\infty$ . The properties of a metric, namely,

- (1)  $\rho(f, g) = 0$  if and only if  $f = g$ ,
- (2)  $\rho(f, g) = \rho(g, f)$ , and
- (3)  $\rho(f, h) \leq \rho(f, g) + \rho(g, h)$ ,

follow immediately from properties (1)–(3) of the norm. It is easily seen that

## 2 Banach Algebra Techniques in Operator Theory

convergence with respect to the metric  $\rho$  is just uniform convergence. An important property of this metric is that  $C(X)$  is complete with respect to it.

**1.2 Proposition.** If  $X$  is a compact Hausdorff space, then  $C(X)$  is a complete metric space.

*Proof* If  $\{f_n\}_{n=1}^{\infty}$  is a Cauchy sequence, then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_{\infty} = \rho(f_n, f_m)$$

for each  $x$  in  $X$ . Hence,  $\{f_n(x)\}_{n=1}^{\infty}$  is a Cauchy sequence of complex numbers for each  $x$  in  $X$ , so we may define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ . We need to show that  $f$  is in  $C(X)$  and that  $\lim_{n \rightarrow \infty} \|f - f_n\|_{\infty} = 0$ . To that end, given  $\varepsilon > 0$ , choose  $N$  such that  $n, m \geq N$  implies  $\|f_n - f_m\|_{\infty} < \varepsilon$ . For  $x_0$  in  $X$  there exists a neighborhood  $U$  of  $x_0$  such that  $|f_N(x_0) - f_N(x)| < \varepsilon$  for  $x$  in  $U$ . Therefore,

$$\begin{aligned} |f(x_0) - f(x)| &\leq \lim_{n \rightarrow \infty} |f_n(x_0) - f_N(x_0)| + \lim_{n \rightarrow \infty} |f_N(x_0) - f_N(x)| \\ &\quad + \lim_{n \rightarrow \infty} |f_N(x) - f_n(x)| \\ &\leq 3\varepsilon \end{aligned}$$

which implies  $f$  is continuous. Further, for  $n \geq N$  and  $x$  in  $X$ , we have

$$\begin{aligned} |f_n(x) - f(x)| &= \left| f_n(x) - \lim_{m \rightarrow \infty} f_m(x) \right| = \lim_{m \rightarrow \infty} |f_n(x) - f_m(x)| \\ &\leq \limsup_{m \rightarrow \infty} \|f_n - f_m\|_{\infty} \leq \varepsilon. \end{aligned}$$

Thus  $\lim_{n \rightarrow \infty} \|f_n - f\|_{\infty} = 0$  and hence  $C(X)$  is complete. ■

We next define the notion of Banach space which abstracts the salient properties of the preceding example. We shall see later in this chapter that every Banach space is isomorphic to a subspace of some  $C(X)$ .

**1.3 Definition.** A Banach space is a complex linear space  $\mathcal{X}$  with a norm  $\|\cdot\|$  satisfying

- (1)  $\|f\| = 0$  if and only if  $f = 0$ ,
- (2)  $\|\lambda f\| = |\lambda| \|f\|$  for  $\lambda$  in  $\mathbb{C}$  and  $f$  in  $\mathcal{X}$ , and
- (3)  $\|f + g\| \leq \|f\| + \|g\|$  for  $f$  and  $g$  in  $\mathcal{X}$ ,

such that  $\mathcal{X}$  is complete in the metric given by this norm.

**1.4 Proposition.** Let  $\mathcal{X}$  be a Banach space. The functions

- $a: \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$  defined  $a(f, g) = f + g$ ,
- $s: \mathbb{C} \times \mathcal{X} \rightarrow \mathcal{X}$  defined  $s(\lambda, f) = \lambda f$ , and
- $n: \mathcal{X} \rightarrow \mathbb{R}^+$  defined  $n(f) = \|f\|$

are continuous.



**Proof** Obvious. ■

**1.5 Directed Sets and Nets** The topology of a metric space can be described in terms of the sequences in it that converge. For more general topological spaces a notion of generalized sequence is necessary. In what follows it will often be convenient to describe a topology in terms of its convergent generalized sequences. Thus we proceed to review for the reader the notion of net.

A directed set  $A$  is a partially ordered set having the property that for each pair  $\alpha$  and  $\beta$  in  $A$  there exists  $\gamma$  in  $A$  such that  $\gamma \geq \alpha$  and  $\gamma \geq \beta$ . A net is a function  $\alpha \rightarrow \lambda_\alpha$  on a directed set. If the  $\lambda_\alpha$  all lie in a topological space  $X$ , then the net is said to converge to  $\lambda$  in  $X$  if for each neighborhood  $U$  of  $\lambda$  there exists  $\alpha_U$  in  $A$  such that  $\lambda_\alpha$  is in  $U$  for  $\alpha \geq \alpha_U$ . Two topologies on a space  $X$  coincide if they have the same convergent nets. Lastly, a topology can be defined on  $X$  by prescribing the convergent nets. For further information concerning nets and subnets, the reader should consult [71].

We now consider the convergence of Cauchy nets in a Banach space.

**1.6 Definition.** A net  $\{f_\alpha\}_{\alpha \in A}$  in a Banach space  $X$  is said to be a Cauchy net if for every  $\varepsilon > 0$ , there exists  $\alpha_0$  in  $A$  such that  $\alpha_1, \alpha_2 \geq \alpha_0$  implies  $\|f_{\alpha_1} - f_{\alpha_2}\| < \varepsilon$ .

**1.7 Proposition.** In a Banach space each Cauchy net is convergent.

**Proof** Let  $\{f_\alpha\}_{\alpha \in A}$  be a Cauchy net in the Banach space  $\mathcal{X}$ . Choose  $\alpha_1$  such that  $\alpha \geq \alpha_1$  implies  $\|f_\alpha - f_{\alpha_1}\| < 1$ . Having chosen  $\{\alpha_k\}_{k=1}^n$  in  $A$ , choose  $\alpha_{n+1} \geq \alpha_n$  such that  $\alpha \geq \alpha_{n+1}$  implies

$$\|f_\alpha - f_{\alpha_{n+1}}\| < \frac{1}{n+1}$$

The sequence  $\{f_{\alpha_n}\}_{n=1}^\infty$  is clearly Cauchy and, since  $\mathcal{X}$  is complete, there exists  $f$  in  $\mathcal{X}$  such that  $\lim_{n \rightarrow \infty} f_{\alpha_n} = f$ .

It remains to prove that  $\lim_{\alpha \in A} f_\alpha = f$ . Given  $\varepsilon > 0$ , choose  $n$  such that  $1/n < \varepsilon/2$  and  $\|f_{\alpha_n} - f\| < \varepsilon/2$ . Then for  $\alpha \geq \alpha_n$  we have

$$\|f_\alpha - f\| \leq \|f_\alpha - f_{\alpha_n}\| + \|f_{\alpha_n} - f\| < 1/n + \varepsilon/2 < \varepsilon. \quad \blacksquare$$

We next consider a general notion of summability in a Banach space which will be used in Chapter 3.

**1.8 Definition.** Let  $\{f_\alpha\}_{\alpha \in A}$  be a set of vectors in the Banach space  $\mathcal{X}$ . Let  $\mathcal{F} = \{F \subset A : F \text{ finite}\}$ . If we define  $F_1 \leq F_2$  for  $F_1 \subset F_2$ , then  $\mathcal{F}$  is a directed set. For each  $F$  in  $\mathcal{F}$ , let  $g_F = \sum_{\alpha \in F} f_\alpha$ . If the net  $\{g_F\}_{F \in \mathcal{F}}$  converges to some  $g$  in  $\mathcal{X}$ , then the sum  $\sum_{\alpha \in A} f_\alpha$  is said to converge and we write  $g = \sum_{\alpha \in A} f_\alpha$ .

**1.9 Proposition.** If  $\{f_\alpha\}_{\alpha \in A}$  is a set of vectors in the Banach space  $\mathcal{X}$  such that  $\sum_{\alpha \in A} \|f_\alpha\|$  converges in the real line  $\mathbb{R}$ , then  $\sum_{\alpha \in A} f_\alpha$  converges in  $\mathcal{X}$ .