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Ronald G. Douglas

Banach Algebra Techniques in Operator Theory

Second Edition

巴拿赫代数在算子理论中的应用 第2版

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Preface to the Second Edition

In the quarter century since the first edition of this book appeared, tremendous development has occurred in operator theory and the topics covered here. However, the new edition remains unchanged except that several mistakes and typographical errors have been corrected. Further, a brief report on the current state of the double-asterisk, open, problems is given along with references. No attempt is made to describe other progress that has been made in the study of Toeplitz operators and related topics nor has the bibliography been updated.

Still, it is hoped that a new generation of students will find useful the introduction to operator theory given here.

College Station, Texas
July 1997

Ronald G. Douglas

Preface to the First Edition

Operator theory is a diverse area of mathematics which derives its impetus and motivation from several sources. It began as did practically all of modern analysis with the study of integral equations at the end of the last century. It now includes the study of operators and collections of operators arising in various branches of physics and mechanics as well as other parts of mathematics and indeed is sufficiently well developed to have a logic of its own. The appearance of several monographs on recent studies in operator theory testifies both to its vigor and breadth.

The intention of this book is to discuss certain advanced topics in operator theory and to provide the necessary background for them assuming only the standard senior-first year graduate courses in general topology, measure theory, and algebra. There is no attempt at completeness and many "elementary" topics are either omitted or mentioned only in the problems. The intention is rather to obtain the main results as quickly as possible.

The book begins with a chapter presenting the basic results in the theory of Banach spaces along with many relevant examples. The second chapter concerns the elementary theory of commutative Banach algebras since these techniques are essential for the approach to operator theory presented in the later chapters. Then after a short chapter on the geometry of Hilbert space, the study of operator theory begins in earnest. In the fourth chapter operators on Hilbert space are studied and a rather sophisticated version of the spectral theorem is obtained. The notion of a C^*-algebra is introduced and used throughout the last half of this chapter. The study of compact operators and Fredholm operators is taken up in the fifth chapter along with certain ancillary results concerning ideals in C^*-algebras. The approach here is a bit unorthodox but is suggested by modern developments.

The last two chapters are of a slightly different character and present a systematic development including recent research of the theory of Toeplitz operators. This latter class of operators has attracted the attention of several mathematicians recently and occurs in several rather diverse contexts.

In the sixth chapter certain topics from the theory of Hardy spaces are developed. The selection is dictated by needs of the last chapter and proofs are based on the techniques obtained earlier in the book. The study of Toeplitz operators is taken up in the seventh chapter. Most of what is known in the scalar case is presented including Widom's result on the connectedness of the spectrum.

At the end of each chapter there are source notes which suggest additional reading along with giving some comments on who proved what and when. Although a reasonable attempt has been made in the latter chapters at citing the appropriate source for important results, omissions have undoubtedly occurred. Moreover, the absence of a reference should not be construed to mean the result is due to the author.

In addition, following each chapter is a large number of problems of varying difficulty. The purposes of these are many: to allow the reader to test his understanding; to indicate certain extensions of the theory which are now accessible; to alert the reader to certain important and related results of which he should be aware along with a hint or a reference for the proof; and to point out certain questions for which the answer is not known. These latter questions are indicated by a double asterisk; a single asterisk indicates a difficult problem.

Stony Brook, New York
August 1971

Ronald G. Douglas

Acknowledgments

This book began as a set of lecture notes for a course given at the University of Michigan in Spring, 1968 and again at SUNY at Stony Brook in the academic year, 1969–1970.

I am indebted to many people in the writing of this book. Most of all I would like to thank Bruce Abrahamse who prepared the original notes and who has been a constant source of suggestions and constructive criticism writing portions of later versions. In addition, I would like to thank many friends and colleagues for their many suggestions and, in particular, Paul Halmos, Carl Pearcy, Pasquale Porcelli, Donald Sarason, and Allen Shields with whom I have learned many of the things presented in this book. Special thanks are due to Berrien Moore III who read and criticized the entire manuscript and to Joyce Lemen, Dorothy Lentz, and Carole Alberghine for typing the various versions of this manuscript. Lastly, I would like to thank the National Science Foundation and the Alfred E. Sloan Foundation for various support during the writing of this book.

Stony Brook, New York
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Ronald G. Douglas

Symbols and Notation

Contents

Chapter 1

Banach Spaces

1.1 We begin by introducing the most representative example of a Banach space. Let X be a compact Hausdorff space and let $C(X)$ denote the set of continuous complex-valued functions on X. For f_1 and f_2 in $C(X)$ and λ a complex number, we define:

(1) $(f_1 + f_2)(x) = f_1(x) + f_2(x)$;
(2) $(\lambda f_1)(x) = \lambda f_1(x)$; and
(3) $(f_1 f_2)(x) = f_1(x) f_2(x)$.

With these operations $C(X)$ is a commutative algebra with identity over the complex field $\mathbb{C}$.

Each function f in $C(X)$ is bounded, since it follows from the fact that f is continuous and X is compact that the range of f is a compact subset of $\mathbb{C}$. Thus the least upper bound of $|f|$ is finite; we call this number the norm of f and denote it by

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

The following properties of the norm are easily verified:

(1) $\|f\|_\infty = 0$ if and only if $f = 0$;
(2) $\|\lambda f\|_\infty = |\lambda| \, \|f\|_\infty$;
(3) $\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty$; and
(4) $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$.

We define a metric ρ on $C(X)$ by $\rho(f, g) = \|f - g\|_\infty$. The properties of a metric, namely,

(1) $\rho(f, g) = 0$ if and only if $f = g$,
(2) $\rho(f, g) = \rho(g, f)$, and
(3) $\rho(f, h) \le \rho(f, g) + \rho(g, h)$,

follow immediately from properties (1)–(3) of the norm. It is easily seen that convergence with respect to the metric ρ is just uniform convergence. An important property of this metric is that $C(X)$ is complete with respect to it.

1.2 Proposition. If X is a compact Hausdorff space, then $C(X)$ is a complete metric space.

Proof If $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, then

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty = \rho(f_n, f_m)$$

for each x in X. Hence, $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence of complex numbers for each x in X, so we may define $f(x) = \lim_{n\to\infty} f_n(x)$. We need to show that f is in $C(X)$ and that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$. To that end, given $\varepsilon > O$, choose N such that $n, m \geq N$ implies $\|f_n - f_m\|_\infty < \varepsilon$. For x_0 in X there exists a neighborhood U of x_0 such that $|f_N(x_0) - f_N(x)| < \varepsilon$ for x in U. Therefore,

$$\begin{aligned}|f(x_0) - f(x)| &\leq \lim_{n\to\infty} |f_n(x_0) - f_N(x_0)| + \lim_{n\to\infty} |f_N(x_0) - f_N(x)| \\ &\quad + \lim_{n\to\infty} |f_N(x) - f_n(x)| \\ &\leq 3\varepsilon\end{aligned}$$

which implies f is continuous. Further, for $n \geq N$ and x in X, we have

$$\begin{aligned}|f_n(x) - f(x)| &= \left|f_n(x) - \lim_{m\to\infty} f_m(x)\right| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \\ &\leq \limsup_{m\to\infty} \|f_n - f_m\|_\infty \leq \varepsilon.\end{aligned}$$

Thus $\lim_{n\to\infty} \|f_x - f\|_\infty = 0$ and hence $C(X)$ is complete. ■

We next define the notion of Banach space which abstracts the salient properties of the preceding example. We shall see later in this chapter that every Banach space is isomorphic to a subspace of some $C(X)$.

1.3 Definition. A Banach space is a complex linear space $\mathscr{X}$ with a norm $\| \ \|$ satisfying

(1) $\|f\| = 0$ if and only if $f = 0$,
(2) $\|\lambda f\| = |\lambda| \, \|f\|$ for λ in $\mathbb{C}$ and f in $\mathscr{X}$, and
(3) $\|f + g\| \leq \|f\| + \|g\|$ for f and g in $\mathscr{X}$,

such that $\mathscr{X}$ is complete in the metric given by this norm.

1.4 Proposition. Let $\mathscr{X}$ be a Banach space. The functions

a: $\mathscr{X} \times \mathscr{X} \to \mathscr{X}$ defined $a(f, g) = f + g$,
s: $\mathbb{C} \times \mathscr{X} \to \mathscr{X}$ defined $s(\lambda, f) = \lambda f$, and
n: $X \to \mathbb{R}^+$ defined $n(f) = \|f\|$

are continuous.

Proof Obvious. ■

1.5 Directed Sets and Nets The topology of a metric space can be described in terms of the sequences in it that converge. For more general topological spaces a notion of generalized sequence is necessary. In what follows it will often be convenient to describe a topology in terms of its convergent generalized sequences. Thus we proceed to review for the reader the notion of net.

A directed set A is a partially ordered set having the property that for each pair α and β in A there exists γ in A such that $\gamma \geq \alpha$ and $\gamma \geq \beta$. A net is a function $\alpha \to \lambda_\alpha$ on a directed set. If the λ_α all lie in a topological space X, then the net is said to converge to λ in X if for each neighborhood U of λ there exists α_U in A such that λ_α is in U for $\alpha \geq \alpha_U$. Two topologies on a space X coincide if they have the same convergent nets. Lastly, a topology can be defined on X by prescribing the convergent nets. For further information concerning nets and subnets, the reader should consult [71].

We now consider the convergence of Cauchy nets in a Banach space.

1.6 Definition. A net $\{f_\alpha\}_{\alpha\in A}$ in a Banach space X is said to be a Cauchy net if for every $\varepsilon > 0$, there exists α_0 in A such that $\alpha_1, \alpha_2 \geq \alpha_0$ implies $\|f_{\alpha_1} - f_{\alpha_2}\| < \varepsilon$.

1.7 Proposition. In a Banach space each Cauchy net is convergent.

Proof Let $\{f_\alpha\}_{\alpha\in A}$ be a Cauchy net in the Banach space $\mathscr{X}$. Choose α_1 such that $\alpha \geq \alpha_1$ implies $\|f_\alpha - f_{\alpha_1}\| < 1$. Having chosen $\{\alpha_k\}_{k=1}^n$ in A, choose $\alpha_{n+1} \geq \alpha_n$ such that $\alpha \geq \alpha_{n+1}$ implies

$$\|f_\alpha - f_{\alpha_{n+1}}\| < \frac{1}{n+1}$$

The sequence $\{f_{\alpha_n}\}_{n=1}^\infty$ is clearly Cauchy and, since $\mathscr{X}$ is complete, there exists f in $\mathscr{X}$ such that $\lim_{n\to\infty} f_{\alpha_n} = f$.

It remains to prove that $\lim_{\alpha\in A} f_\alpha = f$. Given $\varepsilon > 0$, choose n such that $1/n < \varepsilon/2$ and $\|f_{\alpha_n} - f\| < \varepsilon/2$. Then for $\alpha \geq \alpha_n$ we have

$$\|f_\alpha - f\| \leq \|f_\alpha - f_{\alpha_n}\| + \|f_{\alpha_n} - f\| < 1/n + \varepsilon/2 < \varepsilon.$$ ■

We next consider a general notion of summability in a Banach space which will be used in Chapter 3.

1.8 Definition. Let $\{f_\alpha\}_{\alpha\in A}$ be a set of vectors in the Banach space $\mathscr{X}$. Let $\mathscr{F} = \{F \subset A : F \text{ finite}\}$. If we define $F_1 \leq F_2$ for $F_1 \subset F_2$, then $\mathscr{F}$ is a directed set. For each F in $\mathscr{F}$, let $g_F = \sum_{\alpha\in F} f_\alpha$. If the net $\{g_F\}_{F\in\mathscr{F}}$ converges to some g in $\mathscr{X}$, then the sum $\sum_{\alpha\in A} f_\alpha$ is said to converge and we write $g = \sum_{\alpha\in A} f_\alpha$.

1.9 Proposition. If $\{f_\alpha\}_{\alpha\in A}$ is a set of vectors in the Banach space $\mathscr{X}$ such that $\sum_{\alpha\in A} \|f_\alpha\|$ converges in the real line $\mathbb{R}$, then $\sum_{\alpha\in A} f_\alpha$ converges in $\mathscr{X}$.