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Lectures in Logic and Set Theory

Volume 2: Set Theory

GEORGE TOURLAKIS

LECTURES IN LOGIC AND SET THEORY

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Lectures in Logic and Set Theory Volume 2

This two-volume work bridges the gap between introductory expositions of logic or set theory on one hand, and the research literature on the other. It can be used as a text in an advanced undergraduate or beginning graduate course in mathematics, computer science, or philosophy. The volumes are written in a user-friendly conversational lecture style that makes them equally effective for self-study or class use.

Volume 2, on formal (ZFC) set theory, incorporates a self-contained "Chapter 0" on proof techniques so that it is based on formal logic, in the style of Bourbaki. The emphasis on basic techniques will provide the reader with a solid foundation in set theory and sets a context for the presentation of advanced topics such as absoluteness, relative consistency results, two expositions of Gödel's constructible universe, numerous ways of viewing recursion, and a chapter on Cohen forcing.

George Tourlakis is Professor of Computer Science at York University of Ontario.

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To the memory of my parents

Preface

This volume contains the basics of Zermelo-Fraenkel axiomatic set theory. It is situated between two opposite poles: On one hand there are elementary texts that familiarize the reader with the vocabulary of set theory and build set-theoretic tools for use in courses in analysis, topology, or algebra – but do not get into metamathematical issues. On the other hand are those texts that explore issues of current research interest, developing and applying tools (constructibility, absoluteness, forcing, etc.) that are aimed to analyze the inability of the axioms to settle certain set-theoretic questions.

Much of this volume just "does set theory", thoroughly developing the theory of ordinals and cardinals along with their arithmetic, incorporating a careful discussion of diagonalization and a thorough exposition of induction and inductive (recursive) definitions. Thus it serves well those who simply want tools to apply to other branches of mathematics or mathematical sciences in general (e.g., theoretical computer science), but also want to find out about some of the subtler results of modern set theory.

Moreover, a fair amount is included towards preparing the advanced reader to read the research literature. For example, we pay two visits to Gödel's constructible universe, the second of which concludes with a proof of the relative consistency of the axiom of choice and of the generalized continuum hypothesis with ZF. As such a program requires, I also include a thorough discussion of formal interpretations and absoluteness. The lectures conclude with a short but detailed study of Cohen forcing and a proof of the non-provability in ZF of the continuum hypothesis.

The level of exposition is designed to fit a spectrum of mathematical sophistication, from third-year undergraduate to junior graduate level (each group will find here its favourite chapters or sections that serve its interests and level of preparation).

The volume is self-contained. Whatever tools one needs from mathematical logic have been included in Chapter I. Thus, a reader equipped with a combination of sufficient mathematical maturity and patience should be able to read it and understand it. There is a trade-off: the less the maturity at hand, the more the supply of patience must be. To pinpoint this "maturity": At least two courses from among calculus, linear algebra, and discrete mathematics at the junior level should have exposed the reader to sufficient diversity of mathematical issues and proof culture to enable him or her to proceed with reasonable ease.

A word on approach. I use the Zermelo-Fraenkel axiom system *with* the axiom of choice (AC). This is the system known as ZFC. As many other authors do, I simplify nomenclature by allowing "proper classes" in our discussions as part of our metalanguage, but not in the formal language.

I said earlier that this volume contains the "basics". I mean this characterisation in two ways: One, that all the fundamental tools of set theory as needed elsewhere in the mathematical sciences are included in detailed exposition. Two, that I do not present any applications of set theory to other parts of mathematics, because space considerations, along with a decision to include certain advanced relative consistency results, have prohibited this.

"Basics" also entails that I do not attempt to bring the reader up to speed with respect to current research issues. However, a reader who has mastered the advanced metamathematical tools contained here will be able to read the literature on such issues.

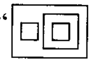
The title of the book reflects two things: One, that all good *short* titles are taken. Two, more importantly, it advertises my conscious effort to present the material in a conversational, user-friendly lecture style. I deliberately employ classroom mannerisms (such as "pauses" and parenthetical "why"s, "what if"s, and attention-grabbing devices for passages that I feel are important). This aims at creating a friendly atmosphere for the reader, especially one who has decided to study the topic without the guidance of an instructor. Friendliness also means steering clear of the terse axiom-definition-theorem recipe, and explaining how some concepts were *arrived at* in their present form. In other words, what makes things tick. Thus, I approach the development of the key concepts of ordinals and cardinals, *initially* and *tentatively*, in the manner they were originally introduced by Georg Cantor (paradox-laden and all). Not only does this afford the reader an understanding of why the modern (von Neumann) approach is superior (and contradiction-free), but it also shows what it tries to accomplish. In the same vein, Russell's paradox is visited no less than three times, leaving us in the end with a firm understanding that it has nothing to do with the "truth" or otherwise of the much-maligned statement "$x \in x$" but it is just the result of a *diagonalization* of the type Cantor originally taught us.

A word on coverage. Chapter I is our "Chapter 0". It contains the tools needed to enable us do our job properly – a bit of mathematical logic, certainly no more than necessary. Chapter II informally outlines what we are about to describe axiomatically: the universe of all the "real" sets and other "objects" of our intuition, a caricature of the von Neumann "universe". It is explained that the whole fuss about axiomatic set theory† is to have a *formal* theory derive true statements about the von Neumann sets, thus enabling us to get to *know* the nature and structure of this universe. If this is to succeed, the chosen axioms must be seen to be "true" in the universe we are describing.

To this end I ensure via *informal discussions* that every axiom that is introduced is seen to "follow" from the principle of the formation of sets by stages, or from some similarly plausible principle devised to keep paradoxes away. In this manner the reader is constantly made aware that we are building a *meaningful* set theory that has relevance to mathematical intuition and expectations (the "real" mathematics), and is not just an artificial choice of a contradiction-free set of axioms followed by the mechanical derivation of a few theorems.

With this in mind, I even make a case for the plausibility of the axiom of choice, based on a popularization of Gödel's constructible universe argument. This occurs in Chapter IV and is informal.

The set theory we do allows *atoms* (or *Urelemente*),‡ just like Zermelo's. The re-emergence of atoms has been defended aptly by Jon Barwise (1975) and others on technical merit, especially when one does "restricted set theories" (e.g., theory of admissible sets).

Our own motivation is not technical; rather it is philosophical and pedagogical. We find it extremely counterintuitive, especially when addressing undergraduate audiences, to tell them that all their familiar mathematical objects – the "stuff of mathematics" in Barwise's words – *are* just perverse "box-in-a-box-in-a-box ... " formations built from an infinite supply of empty boxes. For example, should I be telling my undergraduate students that their *familiar* number "2" *really is* just a short name for something like "  " ? And what will I tell them about "$\sqrt{2}$"?

† O.K., maybe not the *whole* fuss. Axiomatics also allow us to meaningfully ask, and attempt to answer, metamathematical questions of derivability, consistency, relative consistency, independence. But in this volume much of the fuss is indeed about learning set theory.

‡ Allows, but does not insist that there are any.

Some mathematicians have said that set theory (without atoms) speaks only of *sets* and it chooses *not* to speak about objects such as cows or fish (colourful terms for urelements). Well, it does too! Such ("atomless") set theory is known to be perfectly capable of *constructing* "artificial" cows and fish, and can then proceed to talk about such animals as much as it pleases.

While atomless ZFC has the ability to construct or codify all the familiar mathematical objects in it, it does this so well that it betrays the prime directive of the axiomatic method, which is to have a theory that *applies* to diverse concrete (*meta* – i.e., outside the theory and in the realm of "everyday math") mathematical systems. Group theory and projective geometry, for example, fulfill the directive.

In atomless ZFC the opposite appears to be happening: One is asked to *embed* the known mathematics into the formal system.

We prefer a set theory that allows both artificial and real cows and fish, so that when we want to illustrate a point in an example utilizing, say, the everyday set of integers, $\mathbb{Z}$, we can say things like "let the atoms (be interpreted to) include the members of $\mathbb{Z} \ldots$".

But how about technical convenience? Is it not hard to include atoms in a formal set theory? In fact, not at all!

A word on exposition devices. I freely use a pedagogical feature that, I believe, originated in Bourbaki's books – that is, marking an important or difficult topic by placing a "winding road" sign in the margin next to it. I am using here the same symbol that Knuth employed in his TEXbook, namely, ☡, marking with it the beginning and end of an important passage.

Topics that are advanced, or of the "read at your own risk" type, *can be omitted without loss of continuity*. They are delimited by a double sign, ☡☡.

Most chapters end with several exercises. I have stopped making attempts to sort exercises between "hard" and "just about right", as such classifications are rather subjective. In the end, I'll pass on to you the advice one of my professors at the University of Toronto used to offer: "Attempt all the problems. Those you can do, don't do. Do the ones you cannot".

What to read. Just as in the advice above, I suggest that you read everything that you do not already know if time is no object. In a class environment the coverage will depend on class length and level, and I defer to the preferences of the instructor. I suppose that a fourth-year undergraduate audience ought to see the informal construction of the constructible universe in Chapter IV, whereas a graduate audience would rather want to see the formal version in Chapter VI. The latter group will probably also want to be exposed to Cohen forcing.

Acknowledgments. I wish to thank all those who taught me, a group that is too large to enumerate, in which I must acknowledge the presence and influence of my parents, my students, and the writings of Shoenfield (in particular, 1967, 1978, 1971).

The staff at Cambridge University Press provided friendly and expert support, and I thank them. I am particularly grateful for the encouragement received from Lauren Cowles and Caitlin Doggart at the initial (submission and refereeing) and advanced stages (production) of the publication cycle respectively.

I also wish to record my appreciation to Zach Dorsey of TechBooks and his team. In both volumes they tamed my English and LaTeX, fitting them to Cambridge specifications, and doing so with professionalism and flexibility.

This has been a long project that would have not been successful without the support and understanding – for my long leaves of absence in front of a computer screen – that only one's family knows how to provide.

I finally wish to thank Donald Knuth and Leslie Lamport for their typesetting systems TeX and LaTeX that make technical writing fun (and also empower authors to load the pages with ☡ and other signs).

George Tourlakis
Toronto, March 2002

Contents

I

A Bit of Logic: A User's Toolbox

This prerequisite chapter – what some authors call a "Chapter 0" – is an abridged version of Chapter I of volume 1 of my *Lectures in Logic and Set Theory*. It is offered here just in case that volume *Mathematical Logic* is not readily accessible.

Simply put, logic† is about *proofs* or *deductions*. From the point of view of the *user* of the subject – whose best interests we attempt to serve in this chapter – logic ought to be just a toolbox which one can employ to prove theorems, for example, in set theory, algebra, topology, theoretical computer science, etc.

The volume at hand is about an important specimen of a mathematical theory, or logical theory, namely, axiomatic set theory. Another significant example, which we do not study here, is arithmetic. Roughly speaking, a mathematical theory consists on one hand of assumptions that are specific to the subject matter – the so-called *axioms* – and on the other hand a toolbox of logical rules. One usually performs either of the following two activities with a mathematical theory: One may choose to work *within the theory*, that is, employ the tools and the axioms for the sole purpose of proving theorems. Or one can take the entire theory as an *object of study* and study it "from the outside" as it were, in order to pose and attempt to answer questions about the power of the theory (e.g., "does the theory have as theorems all the 'true' statements about the subject matter?"), its reliability (meaning whether it is free from contradictions or not), how its reliability is affected if you add new assumptions (axioms), etc.

Our development of set theory will involve both types of investigations indicated above:

(1) Primarily, we will act as *users* of logic in order to deduce "true" statements about sets (i.e., theorems of set theory) as consequences of certain

† We drop the qualifier "mathematical" from now on, as this is the only type of logic we are about.

"obviously true"† statements that we accept up front without proof, namely, the ZFC axioms.‡ This is pretty much analogous to the behaviour of a geometer whose job is to prove theorems of, say, Euclidean geometry.

(2) We will also look at ZFC from the outside and address some issues of the type "is such and such a sentence (of set theory) provable from the axioms of ZFC and the rules of logic alone?"

It is evident that we need a *precise formulation* of set theory, that is, we must turn it into a *mathematical object* in order to make task (2), above, a meaningful mathematical activity.§ This dictates that we develop logic itself *formally*, and subsequently set theory as a *formal theory*.

Formalism,¶ roughly speaking, is the *abstraction* of the reasoning processes (proofs) achieved by deleting any references to the "truth content" of the component mathematical statements (formulas). What is important in formalist reasoning is solely the syntactic *form* of (mathematical) statements as well as that of the proofs (or deductions) within which these statements appear.

A formalist builds an *artificial language*, that is, an infinite – but *finitely specifiable*# – collection of "words" (meaning *symbol sequences*, also called *expressions*). He|| then uses this language in order to build deductions – that is, *finite sequences* of words – in such a manner that, at each step, he writes down a word if and only if it is "certified" to be *syntactically correct* to do so. "Certification" is granted by a toolbox consisting of the very same rules of logic that we will present in this chapter.

The formalist may pretend, if he so chooses, that the words that appear in a proof are meaningless sequences of meaningless symbols. Nevertheless, such posturing cannot hide the fact that (in any purposefully designed theory) these

† We often quote a word or cluster of related words as a warning that the crude English meaning is not necessarily the intended meaning, or it may be ambiguous. For example, the first "true" in the sentence where this footnote originates is technical, but in a *first approximation* may be taken to mean what "true" means in English. "Obviously true" is an ambiguous term. Obvious to whom? However, the point is – to introduce another ambiguity – that "reasonable people" will accept the truth of the (ZFC) axioms.

‡ This is an acronym reflecting the names of *Zermelo* and *Fraenkel* – the founders of this particular axiomatization – and the fact that the so-called axiom of *choice* is included.

§ Here is an analogy: It is the precision of the rules for the game of chess that makes the notion of analyzing a chessboard configuration meaningful.

¶ The person who practises formalism is a formalist.

The finite specification is achieved by a finite collection of "rules", repeated applications of which build the words.

|| *By definition*, "he", "his", "him" – and their derivatives – are gender-neutral in this volume.

words *codify* "true" (intuitively speaking) statements. Put bluntly, we must have something meaningful to talk about before we bother to codify it.

Therefore, a formal theory is a laboratory version (artificial replica or *simulation*, if you will) of a "real" mathematical theory of the type encountered in mathematics,[†] and formal proofs do unravel (codified versions of) "truths" beyond those embodied in the adopted axioms.

It will be reassuring for the uninitiated that it is a fact of logic that the totality of the "universally true" statements – that is, those that hold in all of mathematics and not only in specific theories – coincides with the totality of statements that we can *deduce purely formally* from some simple universally true *assumptions* such as $x = x$, without any reference to meaning or "truth" (Gödel's completeness theorem for first order logic). In short, in this case formal deducibility is as powerful as "truth". The flip side is that formal deducibility *cannot* be as powerful as "truth" when it is applied to *specific* mathematical theories such as set theory or arithmetic (Gödel's incompleteness theorem).

Formalization allows us to understand the deeper reasons that have prevented set theorists from settling important questions such as the *continuum hypothesis* – that is, the statement that there are no cardinalities between that of the set of natural numbers and that of the set of the reals. This understanding is gathered by "running diagnostics" on our laboratory replica of set theory. That is, just as an engineer evaluates a new airplane design by building and testing a model of the real thing, we can find out, with some startling successes, what are the limitations of our theory, that is, what our assumptions are incapable of logically implying.[‡] If the replica is well built,[§] we can then learn something about the behaviour of the real thing.

In the case of formal set theory and, for example, the question of our failure to resolve the continuum hypothesis, such diagnostics (the methods of Gödel and Cohen – see Chapters VI and VIII) return a simple answer: We have not included enough assumptions in (whether "real" or "formal") set theory to settle this question one way or another.

† Examples of "real" (non-formalized) theories are Euclid's geometry, topology, the theory of groups, and, of course, Cantor's "naïve" or "informal" set theory.

‡ In model theory "model" means exactly the opposite of what it means here. A model airplane abstracts the real thing. A model of a formal (i.e., abstract) theory is a "concrete" or "real" version of the abstract theory.

§ This is where it pays to choose reasonable assumptions, assumptions that are "obviously true".