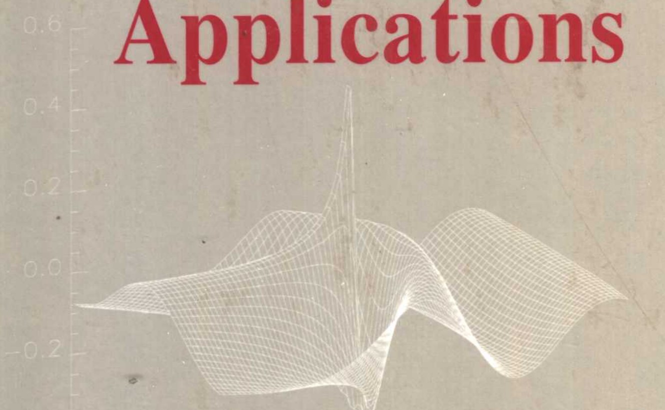
DEAN G. DUFFY

Green's Functions with Applications



STUDIES IN ADVANCED MATHEMATICS

CHAPMAN & HALL/CRC

Green's Functions with Applications

DEAN G. DUFFY

CHAPMAN & HALL/CRC

Boca Raton London New York Washington, D.C.

Library of Congress Cataloging-in-Publication Data

Duffy, Dean G.
Green's functions with applications / Dean Duffy
p. cm. (Studies in advanced mathematics)
Includes bibliographical references and index.
ISBN 1-58488-110-0
1. Green's functions. I. Title. II. Series

QA371 .D74 2001
515'.35—dc21 2001028216
CIP

Catalog record is available from the Library of Congress

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

Visit the CRC Press Web site at www.crcpress.com

International Standard Book Number 1-58488-110-0 (alk. paper)
Library of Congress Card Number 2001028216
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

Introduction

This book had its origin in some electronic mail that I received from William S. Price a number of years ago. He needed to construct a Green's function and asked me if I knew a good book that might assist him. In suggesting several standard texts, I could not help but think that, based on my own experiences utilizing Green's functions alone and in conjunction with numerical solvers, I had my own ideas on how to present this material. It was this thought that ultimately led to the development of this monograph.

The purpose of this book is to provide applied scientists and engineers with a systematic presentation of the various methods available for deriving a Green's function. To this end, I have tried to make this book the most exhaustive source book on Green's function yet available, focusing on every possible analytic technique rather than theory.

After some introductory remarks, the material is classified according to whether we are dealing with an ordinary differential, wave, heat, or Helmholtz equation. Turning first to ordinary differential equations, we have either initial-value or boundary-value problems. After examining initial-value problems, I explore in depth boundary-value problems, both regular and singular. There are essentially two methods: piecing together a solution from the homogeneous solutions, and eigenfunction expansions. Both methods are presented.

Green's functions are particularly well suited for wave problems, as is shown in Chapter 3 with the detailed analysis of electromagnetic waves in surface waveguides and water waves. Before presenting this material, some of the classic solutions in one, two, and three dimensional free space are discussed.

The heat equation and Green's functions have a long association with each other. After discussing heat conduction in free space, the classic solutions of the heat equation in rectangular, cylindrical, and spherical coordinates are offered. The chapter concludes with an interesting application: The application of Green's functions in understanding the stability of fluids and plasmas.

It is not surprising that the final chapter on Poisson's and Helmholtz's equations is the longest. Finding solutions to Poisson's equation gave birth to this technique and Sommerfeld's work at the turn of the twentieth century spurred further development. For each equation, the techniques available for solving them as a function of coordinate system are presented. The final section deals with the computational efficiency of evaluating this class of Green's functions.

This book may be used in a class on boundary-value problems or as a source book for researchers, in which case I recommend that the reader not overlook the problems.

Most books are written with certain assumptions concerning the background of the reader. This book is no exception. The methods for finding Green's functions lean heavily on transform methods because they are particularly well suited for handling the Dirac delta function. For those unfamiliar with using transform methods to solve differential equations, I have summarized these techniques in Appendices A and B. In this sense, the present book is a continuation of my *Transform Methods for Solving Partial Differential Equations*. Because many of the examples and problems involve cylindrical coordinates, an appendix on Bessel functions has been included. In particular, I cover how to find Fourier-Bessel expansions.

A unique aspect of this book is its emphasis on the numerical evaluation of Green's functions. This has taken two particular forms. First, many of the Green's functions that are found in the text and problem sets are illustrated. The motivation here was to assist the reader in developing an intuition about the behavior of Green's functions in certain classes of problem. Second, Green's functions are of little value if they cannot be rapidly computed. Therefore, at several points in the book the question of the computational efficiency and possible methods to accelerate the process have been considered.

Special thanks go to Prof. Michael D. Marcozzi for his many useful suggestions for improving this book. Dr. Tim DelSole provided outstanding guidance in the section on convective/absolute instability. Dr. Chris Linton made several useful suggestions regarding Section 5.8. Finally, I would like to express my appreciation to all those authors and publishers who allowed me the use of their material from the scientific and engineering literature.

Definitions of the Most Commonly Used Functions

Function	Definition
$\delta(t-a)$	$=\begin{cases}\infty, & t=0\\ 0, & t\neq 0\end{cases}, \quad \int_{-\infty}^{\infty}\delta(t)\,dt=1$
$H(t-a)$	$=\begin{cases}1, & t>a,\\ 0, & t<a,\end{cases}, \quad a\geq 0$
$I_n(x)$	modified Bessel function of the first kind and order n
$J_n(x)$	Bessel function of the first kind and order n
$K_n(x)$	modified Bessel function of the second kind and order n
$r_<$	$=\min(r,\rho)$
$r_>$	$=\max(r,\rho)$
$x_<$	$=\min(x,\xi)$
$x_>$	$=\max(x,\xi)$
$Y_n(x)$	Bessel function of the second kind and order n
$y_<$	$=\min(y,\eta)$
$y_>$	$=\max(y,\eta)$
$z_<$	$=\min(z,\zeta)$
$z_>$	$=\max(z,\zeta)$

Contents

Chapter 1
Some Background Material

One of the fundamental problems of field theory[1] is the construction of solutions to linear differential equations when there is a specified source and the differential equation must satisfy certain boundary conditions. The purpose of this book is to show how Green's functions provide a method for obtaining these solutions. In this chapter, some of the mathematical aspects necessary for developing this technique are presented.

1.1 HISTORICAL DEVELOPMENT

In 1828 George Green (1793–1841) published an *Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism*. In this seminal work of mathematical physics, Green sought to determine the electric potential within a vacuum bounded by conductors with specified potentials. In today's notation we would say that he examined the solutions of $\nabla^2 u = -f$ within a volume V that satisfy certain boundary conditions along the boundary S.

---

[1] Any theory in which the basic quantities are fields, such as electromagnetic theory.

In modern notation, Green sought to solve the partial differential equation

$$\nabla^2 g(\mathbf{r}|\mathbf{r}_0) = -4\pi\delta(\mathbf{r}-\mathbf{r}_0), \tag{1.1.1}$$

where $\delta(\mathbf{r}-\mathbf{r}_0)$ is the Dirac delta function. We now know that the solution to (1.1.1) is $g = 1/R$, where $R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$. Although Green recognized the singular nature of g, he proceeded along a different track. First, he proved the theorem that bears his name:

$$\iiint_V \left(\varphi\nabla^2\chi - \chi\nabla^2\varphi\right) dV = \oiint_S (\varphi\nabla\chi - \chi\nabla\varphi)\cdot\mathbf{n}\,dS, \tag{1.1.2}$$

where the outwardly pointing normal is denoted by $\mathbf{n}$ and χ and φ are scalar functions that possess bounded derivatives. Then, by introducing a small ball about the singularity at $\mathbf{r}_0$ because (1.1.2) cannot apply there and then excluding it from the volume V, Green obtained

$$\iiint_V g\nabla^2 u\,dV + \oiint_S g\nabla u\cdot\mathbf{n}\,dS = \iiint_V u\nabla^2 g\,dV + \oiint_S u\nabla g\cdot\mathbf{n}\,dS - 4\pi u(\mathbf{r}_0), \tag{1.1.3}$$

because the surface integral over the small ball is $4\pi u(\mathbf{r}_0)$ as the radius of the ball tends to zero. Next, Green required that *both* g and u satisfy the homogeneous boundary condition $u = 0$ along the surface S. Since $\nabla^2 u = -f$ and $\nabla^2 g = 0$ within V (recall that the point $\mathbf{r}_0$ is excluded from V), he found

$$u(\mathbf{r}) = \frac{1}{4\pi}\oiint_S \overline{u}\,\nabla g\cdot\mathbf{n}\,dS, \tag{1.1.4}$$

when $f = 0$ (Laplace's equation) for any point $\mathbf{r}$ within S, where $\overline{u}$ denotes the value of u on the boundary S. This solved the boundary-value problem once g was found. Green knew that g had to exist; it physically described the electrical potential from a point charge located at $\mathbf{r}_0$.

Green's essay remained relatively unknown until it was published[2] between 1850 and 1854. With its publication the spotlight shifted to the German school of mathematical physics. Although Green himself had not given a name for g, Riemann[3] (1826–1866) would subsequently call

[2] Green, G., 1850, 1852, 1854: An essay on the application of mathematical analysis to the theories of electricity and magnetism. *J. reine angewand. Math.*, **39**, 73–89; **44**, 356–374; **47**, 161–221.

[3] Riemann, B., 1869: *Vorlesungen über die partielle Differentialgleichungen der Physik*, §23; Burkhardt, H., and W. F. Meyer, 1900: *Potentialtheorie* in *Encyklop. d. math. Wissensch.*, **2**, **Part A**, 462–503. See §18.

Figure 1.1.1: Originally drawn to mathematics, Arnold Johannes Wilhelm Sommerfeld (1868–1951) migrated into physics due to Klein's interest in applying the theory of complex variables and other pure mathematics to a range of physical topics from astronomy to dynamics. Later on, Sommerfeld contributed to quantum mechanics and statistical mechanics. (Portrait, AIP Emilio Segrè Visual Archives, Margrethe Bohr Collection.)

it the "Green's function." Then, in 1877, Carl Neumann[4] (1832–1925) embraced the concept of Green's functions in his study of Laplace's equation, particularly in the plane. He found that the two-dimensional equivalent of the Green's function was not described by a singularity of the form $1/|\mathbf{r} - \mathbf{r}_0|$ as in the three-dimensional case but by a singularity of the form $\log(1/|\mathbf{r} - \mathbf{r}_0|)$.

With the function's success in solving Laplace's equation, other equations began to be solved using Green's functions. In the case of

---

[4] Neumann, C., 1877: *Untersuchungen über das Logarithmische und Newton'sche Potential*, Teubner, Leipzig.

the heat equation, Hobson[5] (1856–1933) derived the free-space Green's function for one, two and three dimensions and the French mathematician Appell[6] (1855–1930) recognized that there was a formula similar to Green's for the one-dimensional heat equation. However, it fell to Sommerfeld[7] (1868–1951) to present the modern theory of Green's function as it applies to the heat equation. Indeed, Sommerfeld would be the great champion of Green's functions at the turn of the twentieth century.[8]

The leading figure in the development of Green's functions for the wave equation was Kirchhoff[9] (1824–1887), who used it during his study of the three-dimensional wave equation. Starting with Green's second formula, he was able to show that the three-dimensional Green's function is

$$g(x, y, z, t|\xi, \eta, \zeta, \tau) = \frac{\delta(t - \tau - R/c)}{4\pi R}, \tag{1.1.8}$$

where $R = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ (modern terminology). Although he did not call his solution a Green's function,[10] he clearly grasped the concept that this solution involved a function that we now call the Dirac delta function (see pg. 667 of his *Annalen d. Physik*'s paper). He used this solution to derive his famous *Kirchhoff's theorem*, which is the mathematical expression for Huygen's principle.

The application of Green's function to ordinary differential equations involving boundary-value problems began with the work of Burkhardt[11] (1861–1914). Using results from Picard's theory of ordinary differential equations, he derived the Green's function given by (1.5.35) as well as the properties listed in §2.3. Later on, Bôcher[12] (1867–1918) extended these results to nth order boundary-value problems.

---

[5] Hobson, E. W., 1888: Synthetical solutions in the conduction of heat. *Proc. London Math. Soc.*, **19**, 279–294.

[6] Appell, P., 1892: Sur l'équation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial y} = 0$ et la théorie de la chaleur. *J. Math. pures appl., 4^e série*, **8**, 187–216.

[7] Sommerfeld, A., 1894: Zur analytischen Theorie der Wärmeleitung. *Math. Ann.*, **45**, 263–277.

[8] Sommerfeld, A., 1912: Die Greensche Funktion der Schwingungsgleichung. *Jahresber. Deutschen Math.-Vereinung*, **21**, 309–353.

[9] Kirchhoff, G., 1882: Zur Theorie der Lichtstrahlen. *Sitzber. K. Preuss. Akad. Wiss. Berlin*, 641–669; reprinted a year later in *Ann. Phys. Chem., Neue Folge*, **18**, 663–695.

[10] This appears to have been done by Gutzmer, A., 1895: Über den analytischen Ausdruck des Huygens'schen Princips. *J. reine angewand. Math.*, **114**, 333–337.

[11] Burkhardt, H., 1894: Sur les fonctions de Green relatives a un domaine d'une dimension. *Bull. Soc. Math.*, **22**, 71–75.

[12] Bôcher, M., 1901: Green's function in space of one dimension. *Bull. Amer.*

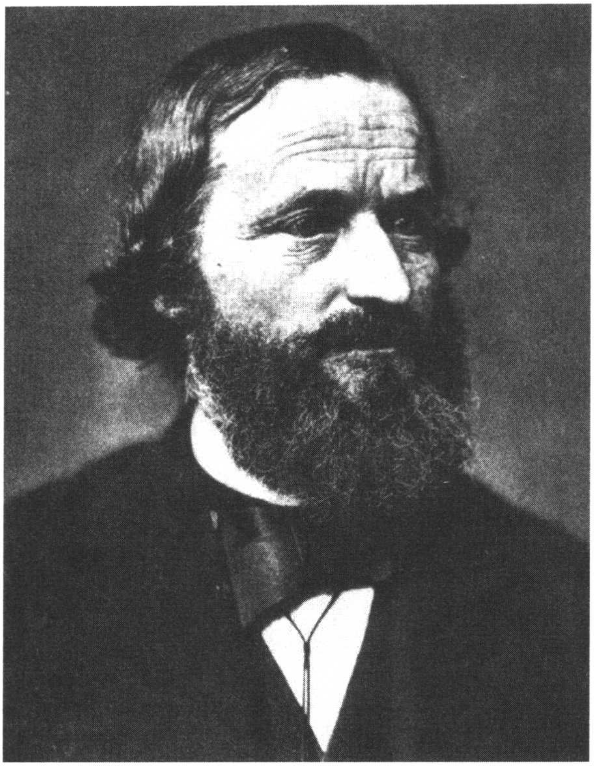


Figure 1.1.2: Gustav Robert Kirchhoff's (1824–1887) most celebrated contributions to physics are the joint founding with Robert Bunsen of the science of spectroscopy, and the discovery of the fundamental law of electromagnetic radiation. Kirchhoff's work on light coincides with his final years as a professor of theoretical physics at Berlin. (Portrait taken from frontispiece of Kirchhoff, G., 1882: *Gesammelte Abhandlungen*. J. A. Barth, 641 pp.)

1.2 THE DIRAC DELTA FUNCTION

Since the 1950s, when Schwartz[13] (1915–) published his theory of distributions, the concept of generalized functions has had an enormous impact on many areas of mathematics, particularly on partial differential equations. In this section, we introduce probably the most important generalized function, the *Dirac delta function*. As we shall shortly see, the entire concept of Green's functions is intimately tied to this most "unusual" function.

Math. Soc., Ser. 2, **7**, 297–299; Bôcher, M., 1911/12: Boundary problems and Green's functions for linear differential and difference equations. *Annals Math., Ser. 2*, **13**, 71–88.

[13] Schwartz, L., 1973: *Théorie des distributions*. Hermann, 418 pp.

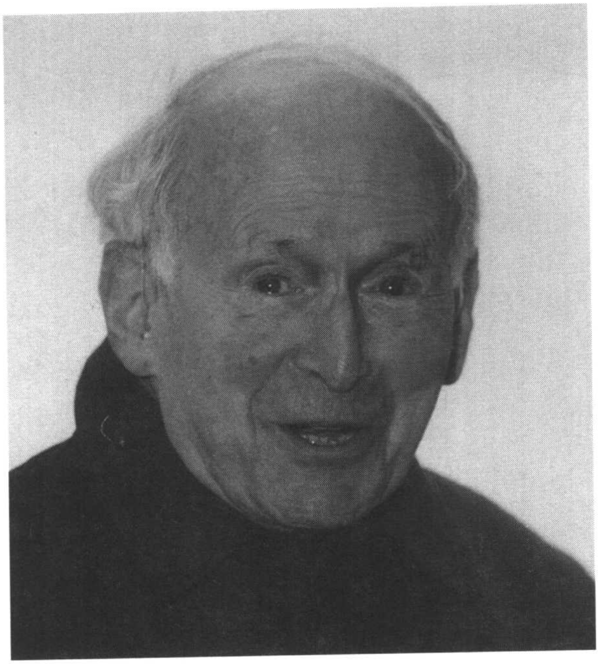


Figure 1.2.1: Laurent Schwartz' (1915–) work on distributions dates from the late 1940s. For this work he was awarded the 1950 Fields medal. (Portrait courtesy of the Ecole Polytechnique, France.)

For many, the Dirac delta function had its birth with the quantum mechanics of Dirac[14] (1902–1984). Modern scholarship[15] has shown, however, that this is simply not true. During the nineteenth century, both physicists and mathematicians used the delta function although physicists viewed it as a purely mathematical idealization that did not exist in nature, while mathematicians used it as an intuitive physical notion without any mathematical reality.

It was the work of Oliver Heaviside (1850–1925) and the birth of electrical engineering that brought the delta function to the attention of the broader scientific and engineering community. In his treatment of a cable that is grounded at both ends, Heaviside[16] introduced the delta function via its sifting property (1.2.9). Consequently, as Laplace transforms became a fundamental tool of electrical engineers, so too did

---

[14] Dirac, P., 1926-7: The physical interpretation of the quantum dynamics. *Proc. R. Soc. London*, **A113**, 621–641.

[15] Lützen, J., 1982: *The Prehistory of the Theory of Distributions*. Springer-Verlag, 232 pp. See chap. 4, part 2.

[16] Heaviside, O., 1950: *Electromagnetic Theory*. Dover Publications, Inc., §267. See Eq. 24.

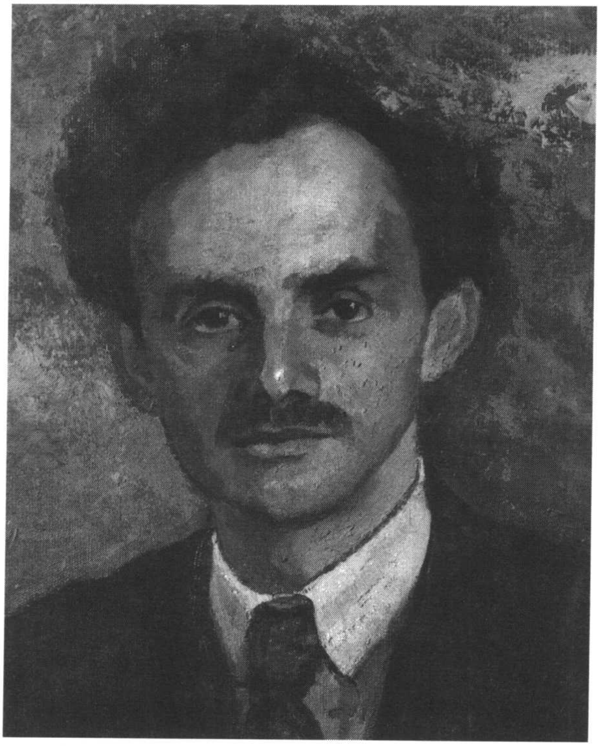


Figure 1.2.2: Paul Adrien Maurice Dirac (1902–1984) ranks among the giants of twentieth-century physics. Awarded the 1933 Nobel Prize in physics for his relativistic quantum mechanics, Dirac employed the delta function during his work on quantum mechanics. In later years, Dirac also helped to formulate Fermi-Dirac statistics and contributed to the quantum theory of electromagnetic radiation. (Portrait reproduced by permission of the President and Council of the Royal Society.)

the use of the delta function.

Despite the delta function's fundamental role in electrical engineering and quantum mechanics, by 1945 there existed several schools of thought concerning its exact nature because Dirac's definition:

$$\delta(t) = \begin{cases} \infty, & t = 0, \\ 0, & t \neq 0, \end{cases} \qquad (\mathbf{1.2.1})$$

such that

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1, \qquad (\mathbf{1.2.2})$$

was unsatisfactory; no conventional function could be found that satisfied (1.2.2).

One approach, especially popular with physicists because it agreed with their physical intuition of a point mass or charge, sought to view the

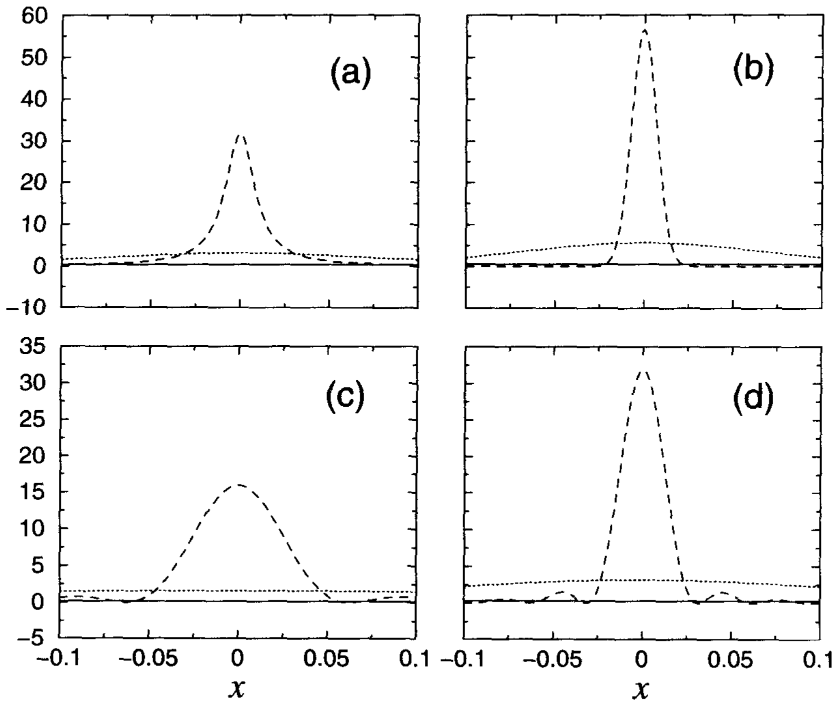


Figure 1.2.3: Frames (a)–(d) illustrate the delta sequences (1.2.4), (1.2.5), $[1 - \cos(nx)]/(n\pi x^2)$, and (1.2.6) as a function of x, respectively. The solid, dotted and dashed lines correspond to $n = 1$, $n = 10$ and $n = 100$, respectively.

delta function as the limit of the sequence of strongly peaked functions $\delta_n(t)$:

$$\delta(t) = \lim_{n\to\infty} \delta_n(t), \qquad \textbf{(1.2.3)}$$

Candidates[17] included

$$\delta_n(t) = \frac{n}{\pi}\frac{1}{1+n^2t^2}, \qquad \textbf{(1.2.4)}$$

$$\delta_n(t) = \frac{n}{\sqrt{\pi}}e^{-n^2t^2}, \qquad \textbf{(1.2.5)}$$

and

$$\delta_n(t) = \frac{1}{n\pi}\frac{\sin^2(nt)}{t^2}. \qquad \textbf{(1.2.6)}$$

The difficulty with this approach was that the limits of these sequences may not exist.

---

[17] Kirchhoff [9] gave (1.2.5) in the limit of $n \to \infty$ as an example of a delta function.