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David Marker

Model Theory: An Introduction

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《国外数学名著系列》(影印版)专家委员会

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格（Springer）出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

In memory of Laura

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Contents

Introduction

Model theory is a branch of mathematical logic where we study mathematical structures by considering the first-order sentences true in those structures and the sets definable by first-order formulas. Traditionally there have been two principal themes in the subject:

- starting with a concrete mathematical structure, such as the field of real numbers, and using model-theoretic techniques to obtain new information about the structure and the sets definable in the structure;
- looking at theories that have some interesting property and proving general structure theorems about their models.

A good example of the first theme is Tarski's work on the field of real numbers. Tarski showed that the theory of the real field is decidable. This is a sharp contrast to Gödel's Incompleteness Theorem, which showed that the theory of the seemingly simpler ring of integers is undecidable. For his proof, Tarski developed the method of quantifier elimination which can be used to show that all subsets of $\mathbb{R}^n$ definable in the real field are geometrically well-behaved. More recently, Wilkie [103] extended these ideas to prove that sets definable in the real exponential field are also well-behaved.

The second theme is illustrated by Morley's Categoricity Theorem, which says that if T is a theory in a countable language and there is an uncountable cardinal κ such that, up to isomorphism, T has a unique model of cardinality κ, then T has a unique model of cardinality λ for every uncountable κ. This line has been extended by Shelah [92], who has developed deep general classification results.

For some time, these two themes seemed like opposing directions in the subject, but over the last decade or so we have come to realize that there are fascinating connections between these two lines. Classical mathematical structures, such as groups and fields, arise in surprising ways when we study general classification problems, and ideas developed in abstract settings have surprising applications to concrete mathematical structures. The most striking example of this synthesis is Hrushovski's [43] application of very general model-theoretic methods to prove the Mordell–Lang Conjecture for function fields.

My goal was to write an introductory text in model theory that, in addition to developing the basic material, illustrates the abstract and applied directions of the subject and the interaction of these two programs.

Chapter 1 begins with the basic definitions and examples of languages, structures, and theories. Most of this chapter is routine, but, because studying definability and interpretability is one of the main themes of the subject, I have included some nontrivial examples. Section 1.3 ends with a quick introduction to $\mathbb{M}^{eq}$. This is a rather technical idea that will not be needed until Chapter 6 and can be omitted on first reading.

The first results of the subject, the Compactness Theorem and the Löwenheim–Skolem Theorem, are introduced in Chapter 2. In Section 2.2 we show that even these basic results have interesting mathematical consequences by proving the decidability of the theory of the complex field. Section 2.4 discusses the back-and-forth method beginning with Cantor's analysis of countable dense linear orders and moving on to Ehrenfeucht–Fraïssé Games and Scott's result that countable structures are determined up to isomorphism by a single infinitary sentence.

Chapter 3 shows how the ideas from Chapter 2 can be used to develop a model-theoretic test for quantifier elimination. We then prove quantifier elimination for the fields of real and complex numbers and use these results to study definable sets.

Chapters 4 and 5 are devoted to the main model-building tools of classical model theory. We begin by introducing types and then study structures built by either realizing or omitting types. In particular, we study prime, saturated, and homogeneous models. In Section 4.3, we show that even these abstract constructions have algebraic applications by giving a new quantifier elimination criterion and applying it to differentially closed fields. The methods of Sections 4.2 and 4.3 are used to study countable models in Section 4.4, where we examine $\aleph_0$-categorical theories and prove Morley's result on the number of countable models. The first two sections of Chapter 5 are devoted to basic results on indiscernibles. We then illustrate the usefulness of indiscernibles with two important applications—a special case of Shelah's Many-Models Theorem in Section 5.3 and the Paris–Harrington independence result in Section 5.4. Indiscernibles also later play an important role in Section 6.5.

Chapter 6 begins with a proof of Morley's Categoricity Theorem in the spirit of Baldwin and Lachlan. The Categoricity Theorem can be thought of as the beginning of modern model theory and the rest of the book is devoted to giving the flavor of the subject. I have made a conscious pedagogical choice to focus on ω-stable theories and avoid the generality of stability, superstability, or simplicity. In this context, forking has a concrete explanation in terms of Morley rank. One can quickly develop some general tools and then move on to see their applications. Sections 6.2 and 6.3 are rather technical developments of the machinery of Morley rank and the basic results on forking and independence. These ideas are applied in Sections 6.4 and 6.5 to study prime model extensions and saturated models of ω-stable theories.

Chapters 7 and 8 are intended to give a quick but, I hope, seductive glimpse at some current directions in the subject. It is often interesting to study algebraic objects with additional model-theoretic hypotheses. In Chapter 7 we study ω-stable groups and show that they share many properties with algebraic groups over algebraically closed fields. We also include Hrushovski's theorem about recovering a group from a generically associative operation which is a generalization of Weil's theorem on group chunks. Chapter 8 begins with a seemingly abstract discussion of the combinatorial geometry of algebraic closure on strongly minimal sets, but we see in Section 8.3 that this geometry has a great deal of influence on what algebraic objects are interpretable in a structure. We conclude with an outline of Hrushovski's proof of the Mordell–Lang Conjecture in one special case.

Because I was trying to write an introductory text rather than an encyclopedic treatment, I have had to make a number of ruthless decisions about what to include and what to omit. Some interesting topics, such as ultraproducts, recursive saturation, and models of arithmetic, are relegated to the exercises. Others, such as modules, the p-adic field, or finite model theory, are omitted entirely. I have also frequently chosen to present theorems in special cases when, in fact, we know much more general results. Not everyone would agree with these choices.

The Reader

While writing this book I had in mind three types of readers:

- graduate students considering doing research in model theory;
- graduate students in logic outside of model theory;
- mathematicians in areas outside of logic where model theory has had interesting applications.

For the graduate student in model theory, this book should provide a firm foundation in the basic results of the subject while whetting the appetite for further exploration. My hope is that the applications given in Chapters 7 and 8 will excite students and lead them to read the advanced texts [7], [18], [76], and [86] written by my friends.

The graduate student in logic outside of model theory should, in addition to learning the basics, get an idea of some of the main directions of the modern subject. I have also included a number of special topics that I think every logician should see at some point, namely the random graph, Ehrenfeucht–Fraïssé Games, Scott's Isomorphism Theorem, Morley's result on the number of countable models, Shelah's Many-Models Theorem, and the Paris–Harrington Theorem.

For the mathematician interested in applications, I have tried to illustrate several of the ways that model theory can be a useful tool in analyzing classical mathematical structures. In Chapter 3, we develop the method of quantifier elimination and show how it can be used to prove results about algebraically closed fields and real closed fields. One of the areas where model-theoretic ideas have had the most fruitful impact is differential algebra. In Chapter 4, we introduce differentially closed fields. Differentially closed fields are very interesting ω-stable structures. Chapters 6, 7, and 8 contain a number of illustrations of the impact of stability-theoretic ideas on differential algebra. In particular, in Section 7.4 we give Poizat's proof of Kolchin's theorem on differential Galois groups of strongly normal extensions. In Chapter 7, we look at classical mathematical objects—groups—under additional model-theoretic assumptions—ω-stability. We also use these ideas to give more information about algebraically closed fields. In Section 8.3, we give an idea of how ideas from geometric model theory can be used to answer questions in Diophantine geometry.

Prerequisites

Chapter 1 begins with the basic definitions of languages and structures. Although a mathematically sophisticated reader with little background in mathematical logic should be able to read this book, I expect that most readers will have seen this material before. The ideal reader will have already taken one graduate or undergraduate course in logic and be acquainted with mathematical structures, formal proofs, Gödel's Completeness and Incompleteness Theorems, and the basics about computability. Shoenfield's *Mathematical Logic* [94] or Ebbinghaus, Flum, and Thomas' *Mathematical Logic* [31] are good references.

I will assume that the reader has some familiarity with very basic set theory, including Zorn's Lemma, ordinals, and cardinals. Appendix A summarizes all of this material. More sophisticated ideas from combinatorial set theory are needed in Chapter 5 but are developed completely in the text.

Many of the applications and examples that we will investigate come from algebra. The ideal reader will have had a year-long graduate algebra course and be comfortable with the basics about groups, commutative rings, and fields. Because I suspect that many readers will not have encountered the algebra of formally real fields that is essential in Section 3.3, I have included this material in Appendix B. Lang's *Algebra* [58] is a good reference for most of the material we will need. Ideally the reader will have also seen some elementary algebraic geometry, but we introduce this material as needed.

Using This Book as a Text

I suspect that in most courses where this book is used as a text, the students will have already seen most of the material in Sections 1.1, 1.2, and 2.1. A reasonable one-semester course would cover Sections 2.2, 2.3, the beginning of 2.4, 3.1, 3.2, 4.1–4.3, the beginning of 4.4, 5.1, 5.2, and 6.1. In a year-long course, one has the luxury of picking and choosing extra topics from the remaining text. My own choices would certainly include Sections 3.3, 6.2–6.4, 7.1, and 7.2.

Exercises and Remarks

Each chapter ends with a section of exercises and remarks. The exercises range from quite easy to quite challenging. Some of the exercises develop important ideas that I would have included in a longer text. I have left some important results as exercises because I think students will benefit by working them out. Occasionally, I refer to a result or example from the exercises later in the text. Some exercises will require more comfort with algebra, computability, or set theory than I assume in the rest of the book. I mark those exercises with a dagger.†

The Remarks sections have two purposes. I make some historical remarks and attributions. With a few exceptions, I tend to give references to secondary sources with good presentations rather than the original source. I also use the Remarks section to describe further results and give suggestions for further reading.

Notation

Most of my notation is standard. I use $A \subseteq B$ to mean that A is a subset of B, and $A \subset B$ means A is a proper subset (i.e., $A \subseteq B$ but $A \neq B$).

If A is a set,

$$A^{<\omega} = \bigcup_{n=1}^{\infty} A^n$$

is the set of all finite sequences from A. I write $\overline{a}$ to indicate a finite sequence $(a_1, \ldots, a_n)$. When I write $\overline{a} \in A$, I really mean $\overline{a} \in A^{<\omega}$.

If A is a set, then $|A|$ is the cardinality of A. The *power set* of A is $\mathcal{P}(A) = \{X : X \subseteq A\}$.

In displays, I sometimes use $\Leftarrow$, $\Rightarrow$ as abbreviations for "implies" and $\Leftrightarrow$ as an abbreviation for "if and only if".

Acknowledgments

My approach to model theory has been greatly influenced by many discussions with my teachers, colleagues, collaborators, students, and friends.

My thesis advisor and good friend, Angus Macintyre, has been the greatest influence, but I would also like to thank John Baldwin, Elisabeth Bouscaren, Steve Buechler, Zoé Chatzidakis, Lou van den Dries, Bradd Hart, Leo Harrington, Kitty Holland, Udi Hrushovski, Masanori Itai, Julia Knight, Chris Laskwoski, Dugald Macpherson, Ken McAloon, Margit Messmer, Ali Nesin, Kobi Peterzil, Anand Pillay, Wai Yan Pong, Charlie Steinhorn, Alex Wilkie, Carol Wood, and Boris Zil'ber for many enlightening conversations and Alan Taylor and Bill Zwicker, who first interested me in mathematical logic.

I would also like to thank John Baldwin, Amador Martin Pizarro, Dale Radin, Kathryn Vozoris, Carol Wood, and particularly Eric Rosen for extensive comments on preliminary versions of this book.

Finally, I, like every model theorist of my generation, learned model theory from two wonderful books, C. C. Chang and H. J. Keisler's *Model Theory* and Gerald Sacks *Saturated Model Theory.* My debt to them for their elegant presentations of the subject will be clear to anyone who reads this book.

1
Structures and Theories

1.1 Languages and Structures

In mathematical logic, we use first-order languages to describe mathematical structures. Intuitively, a structure is a set that we wish to study equipped with a collection of distinguished functions, relations, and elements. We then choose a language where we can talk about the distinguished functions, relations, and elements and nothing more. For example, when we study the ordered field of real numbers with the exponential function, we study the structure $(\mathbb{R}, +, \cdot, \exp, <, 0, 1)$, where the underlying set is the set of real numbers, and we distinguish the binary functions addition and multiplication, the unary function $x \mapsto e^x$, the binary order relation, and the real numbers 0 and 1. To describe this structure, we would use a language where we have symbols for $+, \cdot, \exp, <, 0, 1$ and can write statements such as $\forall x \forall y \ \exp(x) \cdot \exp(y) = \exp(x+y)$ and $\forall x \ (x > 0 \rightarrow \exists y \ \exp(y) = x)$. We interpret these statements as the assertions "$e^x e^y = e^{x+y}$ for all x and y" and "for all positive x, there is a y such that $e^y = x$."

For another example, we might consider the structure $(\mathbb{N}, +, 0, 1)$ of the natural numbers with addition and distinguished elements 0 and 1. The natural language for studying this structure is the language where we have a binary function symbol for addition and constant symbols for 0 and 1. We would write sentences such as $\forall x \exists y \ (x = y + y \ \vee \ x = y + y + 1)$, which we interpret as the assertion that "every number is either even or 1 plus an even number."