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## ADVANCED THEORY AND APPLICATIONS

# PROBABILISTIC STRUCTURAL DYNAMICS
## Advanced Theory and Applications

**Y. K. Lin**
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**McGraw-Hill, Inc.**
New York St. Louis San Francisco Auckland Bogotá
Caracas Lisbon London Madrid Mexico City Milan
Montreal New Delhi San Juan Singapore Sydney Tokyo Toronto

This book was set in Times Roman by Publication Services, Inc.
The editors were B. J. Clark and Margery Luhrs;
the production supervisor was Friederich W. Schulte.
The cover was designed by Carla Bauer.
R. R. Donnelley & Sons Company was printer and binder.

**PROBABILISTIC STRUCTURAL DYNAMICS**
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This book is printed on recycled, acid-free paper
*containing 10% postconsumer waste.*

1 2 3 4 5 6 7 8 9 0 DOC DOC 9 0 9 8 7 6 5 4

ISBN 0-07-038038-4

**Library of Congress Cataloging-in-Publication Data**

Lin, Y. K. (Yu-Kweng), (date).
Probabilistic structural dynamics: advanced theory and applications / Y. K. Lin and G. Q. Cai.
p. cm.
Rev. ed. of: Probabilistic theory of structural dynamics, 1967.
Includes bibliographical references (p. ) and index.
ISBN 0-07-038038-4
1. Structural dynamics. 2. Stochastic processes. I. Cai, G. Q. (Guo-Qiang) II. Lin, Y. K. (Yu-Kweng), (date). Probabilistic theory of structural dynamics. III. Title.
TA654.L5 1995
624.1'7—dc20 94-13102

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# Preface

The field of probabilistic structural dynamics has evolved from infancy in the late 1950s to a mature scientific discipline today. Its applications are found in many branches of engineering—aeronautical, astronautical, civil, mechanical, and others. A comprehensive text on the subject requires a balanced treatment of both the mathematical theory of stochastic processes and structural mechanics. This book is a sequel to *Probabilistic Theory of Structural Dynamics,** written with these goals in mind.

The present volume contains some advanced material not generally available at the time of publication of its predecessor volume; thus it is a supplement in several different areas, the most important being the inclusion of multiplicative random excitations on a dynamical system. More thorough treatments of Markov processes are given in Chapters 4, 5, and 7, including the justification of the Markov model idealization from a physical point of view, and the techniques of exact and approximate solutions, applicable to cases of additive random excitations, multiplicative random excitations, or both. Motion stability of dynamical systems due to multiplicative excitations is considered in Chapter 6. Failures due to excursion of the system response into an unsafe region are treated in Chapter 8, again relying strongly on Markov process modeling. Even though the coverage in the earlier volume is still adequate for linear systems under additive random excitations, more rigorous presentations of spectral analysis are given in Chapters 2 and 3 along with recent applications. Chapter 9 is devoted to random uncertainties of system parameters and initial conditions.

*Referring to Y. K. Lin, *Probabilistic Theory of Structural Dynamics,* McGraw-Hill, New York, 1967; reprint R. E. Krieger, Melbourne, Florida, 1976.

The progress made in this and related technical fields during the past nearly three decades has been tremendous. It is unavoidable that some important contributions are omitted, or only briefly touched upon, if the materials are considered not essential for the development of the main themes in the book. When quoting a reference, preference is given to the original literary source; thus many excellent textbooks are not included in our citations. Attempts are made to keep the exposition of mathematical principles rigorous and yet comprehensible to engineers with a sound background of mechanics. The combination of the present and the 1967 volumes has been used successfully for a sequence of two graduate courses in structural engineering or engineering mechanics at the University of Illinois at Urbana-Champaign and at Florida Atlantic University. The inclusion of many examples in earthquake and wind engineering also makes the texts suitable references for researchers on these subjects. Nevertheless, the present volume is by and large self-contained; therefore, immediate access to the earlier volume is not necessary.

Much of the material in the book was generated from sponsored research during the past twenty years. We are indebted to the sponsors of our research projects, including National Science Foundation, National Aeronautics and Space Administration, Army Research Office, Air Force Office of Scientific Research, Office of Naval Research, and National Center for Earthquake Engineering Research.

It is a pleasure to acknowledge the help received during the preparation of the manuscript. We are indebted to S. T. Ariaratnam of University of Waterloo, Canada, who made valuable suggestions on Chapters 4 and 6, and to our colleague I. Elishakoff, who commented on Chapters 1 through 3. Constructive criticisms provided by the following reviewers on behalf of McGraw-Hill are also gratefully acknowledged: Mircea Grigoriu, Cornell University; Pol Spanos, Rice University; and Y. K. Wen, University of Illinois.

*Y. K. Lin*
*G. Q. Cai*

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# CHAPTER 1

# INTRODUCTION

Probabilistic or stochastic structural dynamics is a subject dealing with uncertainty in the motion of engineering structures. The cause of motion uncertainty may be the unpredictability of excitations, the imperfection or lack of accurate information in the modeling of physical problems, or a combination of these. Mathematically speaking, modeling a dynamic system is equivalent to setting up the governing equations and specifying the initial and boundary conditions. Thus a probabilistic dynamics problem is posed in probabilistic terms about such equations and conditions, and the problem is solved by providing answers to the ensuing motion also in probabilistic terms.

The idea can best be illustrated by a simple example. Shown in Fig. 1.0.1 is a massless pendulum subjected to two random forces: a vertical force $P[1+\xi_1(t)]$ and a horizontal force $P\xi_2(t)$, where $P$ is a constant; $\xi_1(t)$ and $\xi_2(t)$ are random functions of time or stochastic processes. Assuming that the two hinges shown in the figure are frictionless and the linkage between them rigid, the equation of motion for this dynamic system can be obtained by equating the clockwise and counterclockwise moments about the upper hinge to yield

$$c\frac{d}{dt}[l\sin\theta(t)]l\cos\theta(t) + P[1+\xi_1(t)]l\sin\theta(t) = P\xi_2(t)l\cos\theta(t) \quad (1.0.1)$$

where $c$ = damping coefficient, $l$ = length of the pendulum, and $\theta(t)$ is the angular displacement that describes the motion of the pendulum. The uncertainty in $\theta(t)$ may be due to uncertainties in $\xi_1(t)$, $\xi_2(t)$, $c$, $l$, $P$, and the initial condition $\theta(0)$, individually or in any combinations. Some simplifying assumptions are implied when writing equation (1.0.1), so that the equation is rather simple but still adequate to convey the basic idea. These assumptions include, for example, frictionless hinges, a rigid linkage between the hinges, and negligible inertial properties of the

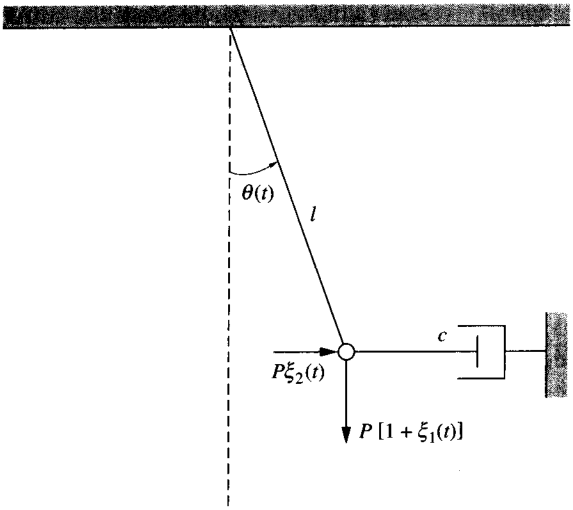


**FIGURE 1.0.1**
A massless pendulum under random excitations.

system. Otherwise, a partial differential equation in both space and time, with higher derivative terms, will be required to describe the system, and the question of uncertainty in other system parameters in the modeling will also arise.

On the other hand, equation (1.0.1) may be simplified further if the angular displacement $\theta(t)$ is small. We then can use the usual approximations $\sin\theta(t) \approx \theta(t)$ and $\cos\theta(t) \approx 1$, reducing the equation to a linear one:

$$\frac{cl}{P}\frac{d}{dt}\theta(t) + [1 + \xi_1(t)]\theta(t) = \xi_2(t) \tag{1.0.2}$$

In the present form, we have lumped the three system parameters into one factor $(cl/P)$, which can be modeled as one random variable. The initial condition $\theta(0)$ may be another random variable, and the two excitations $\xi_1(t)$ and $\xi_2(t)$ two stochastic processes. The problem is posed by specifying the probabilistic or statistical properties of these random variables and stochastic processes, and the solution is obtained in terms of the corresponding properties for $\theta(t)$.

As seen in equation (1.0.1) or (1.0.2), a random excitation may appear in the coefficient of the unknown, or as an inhomogeneous term on the right-hand side of an equation. These two types of random excitations are described as being multiplicative and additive, respectively, referring directly to their positions in a governing equation. They are also called the parametric and external random excitations, respectively, in the literature, referring more to their physical functions. Even though a parametric random excitation may be generated from an outside source, it causes the basic characteristics of the dynamical system to change randomly with time, whereas an external random excitation does not affect the basic characteristics of the system. In particular, a stable system can become unstable, or vice versa, with the presence of multiplicative excitations.

A more complicated dynamical system may be represented by a set of ordinary differential equations of the type

$$\frac{d}{dt}X_j(t) = f_j[\boldsymbol{X}(t), t] + g_{jk}[\boldsymbol{X}(t), t]\xi_k(t) \qquad j = 1, 2, \ldots, N \quad k = 1, 2, \ldots, M \tag{1.0.3}$$

where $\boldsymbol{X}(t) = \{X_1(t), X_2(t), \ldots, X_N(t)\}$ is a vector of system response variables, $\xi_k(t)$ are random excitations, functions $f_j$ and $g_{jk}$ can be either linear or nonlinear, and a repeated subscript in a product indicates a summation. Such a summation convention will be used elsewhere in this book. The class of dynamical systems represented by equation (1.0.3) is actually quite general. It includes continuous structures of finite size which can be discretized using, for example, a Galerkin or finite-element procedure. The first-order form of the differential equations is not a restriction, because a higher order equation can be replaced by an equivalent set of first-order equations. If uncertainties in the modeling of the system itself are not considered, then the functional forms of $f_j$ and $g_{jk}$ are deterministic. A random excitation $\xi_k(t)$ is multiplicative or additive, depending on whether the associated $g_{jk}$ function does or does not depend explicitly on the components of $\boldsymbol{X}(t)$.

Several solution techniques for systems governed by equations of the type of (1.0.3) are discussed in this book. Only the excitations are assumed to be random, except in Chapter 9.

# CHAPTER 2

# SPECTRAL ANALYSIS

Spectral analysis in stochastic dynamics is a generalization of Fourier analysis in deterministic dynamics; therefore, it has the same limitations that the dynamic system must be linear and time-invariant. Under these limitations, functions $f_j$ in equation (1.0.3) must be linear in the components of $\boldsymbol{X}(t)$, and functions $g_{jk}$ must be independent of $\boldsymbol{X}(t)$. In particular, spectral analysis cannot be used to treat a system under multiplicative random excitations even when the system is linear.

When a stochastic dynamics problem is solved using spectral analysis, both the random excitations and the system response variables are expressed as Fourier integrals. In the theory of stochastic processes, differentiation and integration may be defined in terms of convergence in several ways, the most commonly used one being convergence in the mean-square or $L_2$ sense. For convenience, some key results of the $L_2$ calculus are given below. Readers unfamiliar with these convergence concepts are referred to Lin (1967) for details.

Let $X(t)$ be a continuously valued and continuously parametered stochastic process, with an autocorrelation function

$$\phi_{XX}(t_1, t_2) = E[X(t_1)X^*(t_2)] \tag{2.0.1}$$

where $E[\cdot]$ indicates an ensemble average and an asterisk denotes the complex conjugate. We have the following results.

**Continuity.** $X(t)$ is continuous at $t$ in the $L_2$ sense, denoted by

$$\underset{h\to 0}{\text{l.i.m.}}\, X(t+h) = X(t) \tag{2.0.2}$$

if and only if $\phi_{XX}(t_1, t_2)$ is continuous along the diagonal $t_1 = t_2 = t$, where the symbol l.i.m. reads *limit in the mean.*

**Differentiability.** $X(t)$ is differentiable in the $L_2$ sense; that is,

$$\dot{X}(t) = \frac{d}{dt}X(t) = \underset{h\to 0}{\text{l.i.m.}} \frac{X(t+h) - X(t)}{h} \tag{2.0.3}$$

exists, if and only if $(\partial^2/\partial t_1 \partial t_2)\phi_{XX}(t_1, t_2)$ exists along the diagonal $t_1 = t_2 = t$.

**Integrability.** $X(t)$ is Riemann-integrable in the $L_2$ sense; that is,

$$Y(t) = \int_a^b h(t,\tau)X(\tau)d\tau = \underset{\substack{n\to\infty \\ \Delta_n \to 0}}{\text{l.i.m.}} \sum_{j=1}^{n} h(t,\tau_j')X(\tau_j)(\tau_{j+1} - \tau_j) \tag{2.0.4}$$

exists, where $h(t, \tau)$ is a deterministic weighting function, $a = \tau_0 < \tau_1 < \cdots < \tau_{n+1} = b$, $\tau_j \leq \tau_j' \leq \tau_{j+1}$, and $\Delta_n = \max(\tau_{j+1} - \tau_j)$, if and only if

$$I(t) = \int_a^b \int_a^b h(t,\tau)h^*(t,u)\phi_{XX}(\tau,u)\,d\tau\,du < \infty \tag{2.0.5}$$

When applied to the spectral analysis of a stochastic process, the $L_2$ integral in the Riemann form is not entirely satisfactory. Thus we introduce a more general $L_2$ integral in the Stieltjes form as follows:

$$Y(t) = \int_a^b h(t,\tau)\,dZ(\tau) = \underset{\substack{n\to\infty \\ \Delta_n \to 0}}{\text{l.i.m.}} \sum_{j=1}^{n} h(t,\tau_j')[Z(\tau_{j+1}) - Z(\tau_j)] \tag{2.0.6}$$

Equation (2.0.6) reduces to (2.0.4) if $Z(t)$ is differentiable, so that $dZ(\tau) = X(\tau)\,d\tau$. However, (2.0.6) is meaningful even if $Z(t)$ is not differentiable. The $L_2$-Stieltjes integral (2.0.6) exists if and only if

$$I(t) = \int_a^b \int_a^b h(t,\tau)h^*(t,u)E[dZ(\tau)\,dZ^*(u)] < \infty \tag{2.0.7}$$

where $$E[dZ(\tau)\,dZ^*(u)] = E\{[Z(\tau + d\tau) - Z(\tau)][Z^*(u+du) - Z^*(u)]\} \tag{2.0.8}$$

**Commutability of $L_2$ limit and ensemble averaging.** If an $L_2$ limit exists, then the order in which the limit and the ensemble average are taken can be interchanged. For example, if $(d/dt)X(t)$ exists as an $L_2$ derivative, then

$$E\left\{\left[\frac{d}{dt_1}X(t_1)\right]\left[\frac{d}{dt_2}X^*(t_2)\right]\right\} = \frac{\partial^2}{\partial t_1 \partial t_2}E[X(t_1)X^*(t_2)] \tag{2.0.9}$$

or

$$\phi_{\dot{X}\dot{X}}(t_1,t_2) = E[\dot{X}(t_1)\dot{X}^*(t_2)] = \frac{\partial^2}{\partial t_1 \partial t_2}\phi_{XX}(t_1,t_2) \tag{2.0.10}$$

Therefore, the autocorrelation function of the derivative process $\dot{X}(t)$ can be obtained as the mixed second derivative of the autocorrelation function $\phi_{XX}(t_1, t_2)$ of the original process $X(t)$. Moreover,

$$E[\dot{X}^2(t)] = \left[\frac{\partial^2}{\partial t_1 \partial t_2}\phi_{XX}(t_1,t_2)\right]_{t_1=t_2=t} \tag{2.0.11}$$

The boundedness of the right-hand side of (2.0.11) is precisely the necessary and sufficient condition for the existence of $\dot{X}(t)$.

It can also be shown that the necessary and sufficient condition (2.0.5) for the $L_2$-Riemann integral $Y(t)$, defined in (2.0.4), amounts to requiring that the mean-square value $E[Y^2(t)]$ be finite, and that the autocorrelation of $Y(t)$ can be obtained from

$$\phi_{YY}(t_1, t_2) = \int_a^b \int_a^b h(t_1, \tau)h^*(t_2, u)\phi_{XX}(\tau, u)\, d\tau\, du \tag{2.0.12}$$

The case of the $L_2$-Stieltjes integral is similar.

## 2.1 STOCHASTIC PROCESSES WITH UNCORRELATED AND ORTHOGONAL INCREMENTS

To explain several subtle points in the Fourier-Stieltjes integral representation of a stochastic process, we require two fundamental concepts.

Let $Z(\omega)$ be a complex-valued stochastic process, defined on $a \le \omega \le b$, satisfying

$$E[|Z(\omega_2) - Z(\omega_1)|^2] < \infty \qquad a \le \omega_1, \omega_2 \le b \tag{2.1.1}$$

**Definition.** $Z(\omega)$ is said to be a stochastic process with uncorrelated increments if

$$E\{[Z(\omega_2) - Z(\omega_1)][Z^*(\omega_4) - Z^*(\omega_3)]\} = E[Z(\omega_2) - Z(\omega_1)]E[Z^*(\omega_4) - Z^*(\omega_3)] \tag{2.1.2}$$

for any nonoverlapping intervals $(\omega_1, \omega_2]$ and $(\omega_3, \omega_4]$, where $a \le \omega_1 < \omega_2 \le \omega_3 < \omega_4 \le b$, and $(\omega_j, \omega_k]$ denotes an interval $\omega_j < \omega \le \omega_k$ which includes point $\omega_k$ but not point $\omega_j$. This common practice of representing a closed end of an interval by a bracket and an open end by a parenthesis is followed hereafter.

**Definition.** $Z(\omega)$ is said to be a stochastic process with orthogonal increments if it has uncorrelated increments, and if the right-hand side of (2.1.2) is equal to zero.

It is clear from the preceding definition that if $Z(\omega)$ is a stochastic process with orthogonal increments, then $Y(\omega) = Z(\omega) - Z_0$ is also a stochastic process with orthogonal increments, where $Z_0$ is an arbitrary random variable and $E[|Z_0|^2]$ is finite. In particular, $Z_0$ may be chosen as $Z(\omega_0)$, where $\omega_0$ is an arbitrary reference point on the $\omega$ axis.

It can easily be shown that if $Z(\omega)$ is a stochastic process with orthogonal increments, then

$$E\{[Z(\omega_3) - Z(\omega_1)][Z^*(\omega_4) - Z^*(\omega_2)]\} = E\{|Z(\omega_3) - Z(\omega_2)|^2\} \tag{2.1.3}$$

$$a \le \omega_1 < \omega_2 \le \omega_3 < \omega_4 \le b$$

## 2.2 SPECTRAL REPRESENTATION OF A CORRELATION-STATIONARY STOCHASTIC PROCESS

Let $Z(\omega)$ be a stochastic process with orthogonal increments, and let $\omega_0$ be an arbitrarily chosen reference point on the $\omega$ axis. Define a deterministic function

$$\Psi(\omega) = \begin{cases} E\{|Z(\omega) - Z(\omega_0)|^2\} & \omega \geq \omega_0 \\ -E\{|Z(\omega_0) - Z(\omega)|^2\} & \omega < \omega_0 \end{cases} \tag{2.2.1}$$

which implies that $\Psi(\omega_0) = 0$. It can be shown that if $\omega_2 \geq \omega_1$, then regardless of the choice of $\omega_0$,

$$E\{|Z(\omega_2) - Z(\omega_1)|^2\} = \Psi(\omega_2) - \Psi(\omega_1) \tag{2.2.2}$$

We shall now prove that (2.2.2) is valid when $\omega_0 < \omega_1 \leq \omega_2$. Note that the left-hand side of (2.2.2) is equal to

$$\begin{aligned} &E\{[Z(\omega_2) - Z(\omega_1)][Z^*(\omega_2) - Z^*(\omega_1)]\} \\ &\quad = E(\{[Z(\omega_2) - Z(\omega_0)] - [Z(\omega_1) - Z(\omega_0)]\}\{[Z^*(\omega_2) - Z^*(\omega_0)] \\ &\quad - [Z^*(\omega_1) - Z^*(\omega_0)]\}) \end{aligned} \tag{2.2.3}$$

Upon expanding the right-hand side of (2.2.3) and using (2.1.3) and (2.2.1), we obtain

$$\text{rhs} = \Psi(\omega_2) - \Psi(\omega_1) - \Psi(\omega_1) + \Psi(\omega_1) = \Psi(\omega_2) - \Psi(\omega_1) \tag{2.2.4}$$

which agrees with the right-hand side of (2.2.2). The cases of $\omega_1 < \omega_0 < \omega_2$ and $\omega_1 < \omega_2 < \omega_0$ are left as exercises for the reader.

Since the left-hand side of (2.2.2) is nonnegative, $\Psi(\omega)$ must be nondecreasing. In particular, letting $\omega_1 = \omega$, $\omega_2 = \omega + d\omega$ in (2.2.2), we obtain

$$E\{|dZ(\omega)|^2\} = d\Psi(\omega) \tag{2.2.5}$$

Some comments on the result (2.2.5) are in order:

1. If $\Psi(\omega)$ is not differentiable at some $\omega_j$, then $d\Psi(\omega_j)$ can be finite. In this case, $dZ(\omega_j)$ is also finite.
2. If $\Psi(\omega)$ is differentiable, that is, $d\Psi(\omega) = O(d\omega)$, where $O(\cdot)$ denotes the order of magnitude of the quantity within the parentheses, then $dZ(\omega) = O(\sqrt{d\omega})$.

Therefore, a stochastic process with orthogonal increments is always *not* differentiable.

The definition given in Section 2.1 for an orthogonal-increment process is not readily useful in subsequent applications. We next develop an alternative equivalent