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Glen E. Bredon

Topology and Geometry

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Preface

The golden age of mathematics—that was not the age of Euclid, it is ours.

C.J. KEYSER

This time of writing is the hundredth anniversary of the publication (1892) of Poincaré's first note on topology, which arguably marks the beginning of the subject of algebraic, or "combinatorial," topology. There was earlier scattered work by Euler, Listing (who coined the word "topology"), Möbius and his band, Riemann, Klein, and Betti. Indeed, even as early as 1679, Leibniz indicated the desirability of creating a geometry of the topological type. The establishment of topology (or "analysis situs" as it was often called at the time) as a coherent theory, however, belongs to Poincaré.

Curiously, the beginning of general topology, also called "point set topology," dates fourteen years later when Fréchet published the first abstract treatment of the subject in 1906.

Since the beginning of time, or at least the era of Archimedes, smooth manifolds (curves, surfaces, mechanical configurations, the universe) have been a central focus in mathematics. They have always been at the core of interest in topology. After the seminal work of Milnor, Smale, and many others, in the last half of this century, the topological aspects of smooth manifolds, as distinct from the differential geometric aspects, became a subject in its own right. While the major portion of this book is devoted to algebraic topology, I attempt to give the reader some glimpses into the beautiful and important realm of smooth manifolds along the way, and to instill the tenet that the algebraic tools are primarily intended for the understanding of the geometric world.

This book is intended as a textbook for a beginning (first-year graduate) course in algebraic topology with a strong flavoring of smooth manifold theory. The choice of topics represents the ideal (to the author) course. In practice, however, most such courses would omit many of the subjects in the book. I would expect that most such courses would assume previous knowledge of general topology and so would skip that chapter, or be limited to a brief run-through of the more important parts of it. The section on homotopy should be covered, however, at some point. I do not go deeply into general topology, but I do believe that I cover the subject as completely as a mathematics student needs unless he or she intends to specialize in that area.

It is hoped that at least the introductory parts of the chapter on differentiable manifolds will be covered. The first section on the Implicit Function Theorem might best be consigned to individual reading. In practice, however, I expect that chapter to be skipped in many cases with that material assumed covered in another course in differential geometry, ideally concurrent. With that possibility in mind, the book was structured so that that material is not essential to the remainder of the book. Those results that use the methods of smooth manifolds and that are crucial to other parts of the book are given separate treatment by other methods. Such duplication is not so large as to be consumptive of time, and, in any case, is desirable from a pedagogic standpoint. Even the material on differential forms and de Rham's Theorem in the chapter on cohomology could be omitted with little impact on the other parts of the book. That would be a great shame, however, since that material is of such interest on its own part as well as serving as a motivation for the introduction of cohomology. The section on the de Rham theory of $\mathbf{CP}^n$ could, however, best be left to assigned reading. Perhaps the main use of the material on differentiable manifolds is its impact on examples and applications of algebraic topology.

As is common practice, the starred sections are those that could be omitted with minimal impact on other nonstarred material, but the starring should not be taken as a recommendation for that aim. In some cases, the starred sections make more demands on mathematical maturity than the others and may contain proofs that are more sketchy than those elsewhere.

This book is not intended as a source book. There is no attempt to present material in the most general form, unless that entails no expense of time or clarity. Exceptions are cases, such as the proof of de Rham's Theorem, where generality actually improves both efficiency and clarity. Treatment of esoteric byways is inappropriate in textbooks and introductory courses. Students are unlikely to retain such material, and less likely to ever need it, if, indeed, they absorb it in the first place.

As mentioned, some important results are given more than one proof, as much for pedagogic reasons as for maintaining accessibility of results essential to algebraic topology for those who choose to skip the geometric treatments of those results. The Fundamental Theorem of Algebra is given no less than four topological proofs (in illustration of various results). In places where choice is necessary between competing approaches to a given topic, preference has been given to the one that leads to the best understanding and intuition.

In the case of homology theory, I first introduce singular homology and derive its simpler properties. Then the axioms of Eilenberg, Steenrod, and Milnor are introduced and used exclusively to derive the computation of the homology groups of cell complexes. I believe that doing this from the

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axioms, without recourse to singular homology, leads to a better grasp of the functorial nature of the subject. (It also provides a uniqueness proof gratis.) This also leads quickly to the major applications of homology theory. After that point, the difficult and technical parts of showing that singular homology satisfies the axioms are dealt with.

Cohomology is introduced by first treating differential forms on manifolds, introducing the de Rham cohomology and then linking it to singular homology. This leads naturally to singular cohomology. After development of the simple properties of singular cohomology, de Rham cohomology is returned to and de Rham's famous theorem is proved. (This is one place where treatment of a result in generality, for all differentiable manifolds and not just compact ones, actually provides a simpler and cleaner approach.)

Appendix B contains brief background material on "naive" set theory. The other appendices contain ancillary material referred to in the main text, usually in reference to an inessential matter.

There is much more material in this book than can be covered in a one-year course. Indeed, if everything is covered, there is enough for a two-year course. As a suggestion for a one-year course, one could start with Chapter II, assigning Section 1 as individual reading and then covering Sections 2 through 11. Then pick up Section 14 of Chapter I and continue with Chapter III, Sections 1 through 8, and possibly Section 9. Then take Chapter IV except for Section 12 and perhaps omitting some details about CW-complexes. Then cover Chapter V except for the last three sections. Finally, Chapter VI can be covered through Section 10. If there is time, coverage of Hopf's Theorem in Section 11 of Chapter V is recommended. Alternatively to the coverage of Chapter VI, one could cover as much of Chapter VII as is possible, particularly if there is not sufficient time to reach the duality theorems of Chapter VI.

Although I do make occasional historical remarks, I make no attempt at thoroughness in that direction. An excellent history of the subject can be found in Dieudonné [1]. That work is, in fact, much more than a history and deserves to be in every topologist's library.

Most sections of the book end with a group of problems, which are exercises for the reader. Some are harder, or require more "maturity," than others and those are marked with a ◆. Problems marked with a ⟡ are those whose results are used elsewhere in the main text of the book, explicitly or implicitly.

Glen E. Bredon

Acknowledgments

It was perfect, it was rounded, symmetrical, complete, colossal.

MARK TWAIN

Unlike the object of Mark Twain's enthusiasm, quoted above (and which has no geometric connection despite the four geometric–topological adjectives), this book is far from perfect. It is simply the best I could manage. My deepest thanks go to Peter Landweber for reading the entire manuscript and for making many corrections and suggestions. Antoni Kosinski also provided some valuable assistance. I also thank the students in my course on this material in the spring of 1992, and previous years, Jin-Yen Tai in particular, for bringing a number of errors to my attention and for providing some valuable pedagogic ideas.

Finally, I dedicate this book to the memory of Deane Montgomery in deep appreciation for his long-term support of my work and of that of many other mathematicians.

Glen E. Bredon

Contents

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CHAPTER I
General Topology

A round man cannot be expected to fit in a square hole right away. He must have time to modify his shape.

MARK TWAIN

1. Metric Spaces

We are all familiar with the notion of distance in euclidean n-space: If $\mathbf{x}$ and $\mathbf{y}$ are points in $\mathbf{R}^n$ then

$$\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}.$$

This notion of distance permits the definition of continuity of functions from one euclidean space to another by the usual ϵ–δ definition:

$$f\colon \mathbf{R}^n \to \mathbf{R}^k \text{ is continuous at } \mathbf{x} \in \mathbf{R}^n \text{ if, given } \epsilon > 0,$$
$$\exists \delta > 0 \ni \operatorname{dist}(\mathbf{x}, \mathbf{y}) < \delta \quad \Rightarrow \quad \operatorname{dist}(f(\mathbf{x}), f(\mathbf{y})) < \epsilon.$$

Although the spaces of most interest to us in this book are subsets of euclidean spaces, it is useful to generalize the notion of "space" to get away from such a hypothesis, because it would be very complicated to try to verify that spaces we construct are always of this type. In topology, the central notion is that of continuity. Thus it would usually suffice for us to treat "spaces" for which we can give a workable definition of continuity.

We could define continuity as above for any "space" which has a suitable notion of distance. Such spaces are called "metric spaces."

1.1. Definition. A *metric space* is a set X together with a function

$$\operatorname{dist}\colon X \times X \to \mathbf{R},$$

called a *metric*, such that the following three laws are satisfied:

(1) (positivity) $\operatorname{dist}(x, y) \geq 0$ with equality $\Leftrightarrow x = y$;
(2) (symmetry) $\operatorname{dist}(x, y) = \operatorname{dist}(y, x)$; and
(3) (triangle inequality) $\operatorname{dist}(x, z) \leq \operatorname{dist}(x, y) + \operatorname{dist}(y, z)$.

In a metric space X we define the "ϵ-ball," $\epsilon > 0$, about a point $x \in X$ to be

$$\boxed{B_\epsilon(x) = \{y \in X \mid \operatorname{dist}(x, y) < \epsilon\}.}$$

Also, a subset $U \subset X$ is said to be "open" if, for each point $x \in U$, there is an ϵ-ball about x completely contained in U. A subset is said to be "closed" if its complement is open. If $y \in B_\epsilon(x)$ and if $\delta = \epsilon - \operatorname{dist}(x, y)$ then $B_\delta(y) \subset B_\epsilon(x)$ by the triangle inequality. This shows that all ϵ-balls are open sets.

It turns out that, for metric spaces, continuity can be expressed completely in terms of open sets:

1.2. Proposition. *A function $f: X \to Y$ between metric spaces is continuous $\Leftrightarrow$ $f^{-1}(U)$ is open in X for each open subset U of Y.*

PROOF. If f is continuous and $U \subset Y$ is open and $f(x) \in U$ then there is an $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$. By continuity, there is a $\delta > 0$ such that f maps the δ-ball about x into $B_\epsilon(f(x))$. This means that $B_\delta(x) \subset f^{-1}(U)$. This implies that $f^{-1}(U)$ is open.

Conversely, suppose $f(x) = y$ and that $\epsilon > 0$ is given. By hypothesis, $f^{-1}(B_\epsilon(y))$ is open and contains x. Therefore, by the definition of an open set, there is a $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(y))$. It follows that if $\operatorname{dist}(x, x') < \delta$ then $f(x') \in B_\epsilon(y)$, and so $\operatorname{dist}(f(x), f(x')) < \epsilon$, proving continuity in the ϵ–δ sense. $\square$

The only examples of metric spaces we have discussed are euclidean spaces and, of course, subsets of those. Even with those, however, there are other reasonable metrics:

$$\operatorname{dist}_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|,$$

$$\operatorname{dist}_3(\mathbf{x}, \mathbf{y}) = \max(|x_i - y_i|).$$

It is not hard to verify, from the following proposition, that these three metrics give the same open sets, and so behave identically with respect to continuity (for maps into or out of them).

1.3. Proposition. *If* dist_1 *and* dist_2 *are metrics on the same set X which satisfy the hypothesis that for any point $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that*

$$\operatorname{dist}_1(x, y) < \delta \quad \Rightarrow \quad \operatorname{dist}_2(x, y) < \epsilon,$$

and

$$\operatorname{dist}_2(x, y) < \delta \quad \Rightarrow \quad \operatorname{dist}_1(x, y) < \epsilon,$$

then these metrics define the same open sets in X.

PROOF. The proof is an easy exercise in the definition of open sets and is left to the reader. $\square$

PROBLEMS

1. Consider the set X of all continuous real valued functions on $[0, 1]$. Show that
$$\operatorname{dist}(f, g) = \int_0^1 |f(x) - g(x)|\, dx$$
defines a metric on X. Is this still the case if continuity is weakened to integrability?

2. If X is a metric space and x_0 is a given point in X, show that the function $f: X \to \mathbf{R}$ given by $f(x) = \operatorname{dist}(x, x_0)$ is continuous.

3. If A is a subset of a metric space X then define a real valued function d on X by $d(x) = \operatorname{dist}(x, A) = \inf\{\operatorname{dist}(x, y) \mid y \in A\}$. Show that d is continuous. (*Hint*: Use the triangle inequality to show that $|d(x_1) - d(x_2)| \leq \operatorname{dist}(x_1, x_2)$.)

2. Topological Spaces

Although most of the spaces that will interest us in this book are metric spaces, or can be given the structure of metric spaces, we will usually only care about continuity of mappings and not the metrics themselves. Since continuity can be expressed in terms of open sets alone, and since some constructions of spaces of interest to us do not easily yield to construction of metrics on them, it is very useful to discard the idea of metrics and to abstract the basic properties of open sets needed to talk about continuity. This leads us to the notion of a general "topological space."

2.1. Definition. A *topological space* is a set X together with a collection of subsets of X called "*open*" sets such that:

(1) the intersection of two open sets is open;
(2) the union of any collection of open sets is open; and
(3) the empty set $\varnothing$ and whole space X are open.

Additionally, a subset $C \subset X$ is called "*closed*" if its complement $X - C$ is open.

Topological spaces are much more general than metric spaces and the range of difference between them and metric spaces is much wider than that between metric spaces and subspaces of euclidean space. For example, it is possible to talk about convergence of sequences of points in metric spaces with little difference from sequences of real numbers. Continuity of functions can be described in terms of convergence of sequences in metric spaces. One can also talk about convergence of sequences in general topological spaces but that no longer is adequate to describe continuity (as we shall see later). Thus it is necessary to exercise care in developing the theory of general topological spaces. We now begin that development, starting with some further basic definitions.

2.2. Definition. If X and Y are topological spaces and $f: X \to Y$ is a function, then f is said to be *continuous* if $f^{-1}(U)$ is open for each open set $U \subset Y$. A *map* is a continuous function.

Since closed sets are just the complements of open sets and since inverse images preserve complements (i.e., $f^{-1}(Y - B) = X - f^{-1}(B)$), it follows that a function $f: X \to Y$ is continuous $\Leftrightarrow f^{-1}(F)$ is closed for each closed set $F \subset Y$.

2.3. Definition. If X is a topological space and $x \in X$ then a set N is called a *neighborhood* of x in X if there is an open set $U \subset N$ with $x \in U$.

Note that a neighborhood is not necessarily an open set, and, even though one usually thinks of a neighborhood as "small," it need not be: the entire space X is a neighborhood of each of its points.

Note that the intersection of any two neighborhoods of x in X is a neighborhood of x, which follows from the axiom that the intersection of two open sets is open.

The intuitive notion of "smallness" of a neighborhood is given by the concept of a neighborhood basis at a point:

2.4. Definition. If X is a topological space and $x \in X$ then a collection $\mathbf{B}_x$ of subsets of X containing x is called a *neighborhood basis* at x in X if each neighborhood of x in X contains some element of $\mathbf{B}_x$ and each element of $\mathbf{B}_x$ is a neighborhood of x.

Neighborhood bases are sometimes convenient in proving functions to be continuous:

2.5. Definition. A function $f: X \to Y$ between topological spaces is said to be *continuous at* x, where $x \in X$, if, given any neighborhood N of $f(x)$ in Y, there is a neighborhood M of x in X such that $f(M) \subset N$.

Since $f(f^{-1}(N)) \subset N$, this is the same as saying that $f^{-1}(N)$ is a neighborhood of x, for each neighborhood N of $f(x)$. Clearly, this need only be checked for N belonging to some neighborhood basis at $f(x)$.

2.6. Proposition. *A function $f: X \to Y$ between topological spaces is continuous $\Leftrightarrow$ it is continuous at each point $x \in X$.*

PROOF. Suppose that f is continuous, i.e., that $f^{-1}(U)$ is open for each open $U \subset Y$. Let N be a neighborhood of $f(x)$ in Y and let U be an open set such that $f(x) \in U \subset N$ as guaranteed by the definition of neighborhood. Then $x \in f^{-1}(U) \subset f^{-1}(N)$ and $f^{-1}(U)$ is open. It follows that $f^{-1}(N)$ is a neighborhood of x. Thus f is continuous at x.

Conversely, suppose that f is continuous at each point and let $U \subset Y$ be an open set. For any $x \in f^{-1}(U)$, $f^{-1}(U)$ is then a neighborhood of x. Thus there exists an open set V_x in X with $x \in V_x \subset f^{-1}(U)$. Hence $f^{-1}(U)$ is the union of the sets V_x for x ranging over $f^{-1}(U)$. Since the union of any collection of open sets is open, it follows that $f^{-1}(U)$ is open. But U was an arbitrary open set in Y and, consequently, f is continuous. □

2.7. Definition. A function $f: X \to Y$ between topological spaces is called a *homeomorphism* if $f^{-1}: Y \to X$ exists (i.e., f is one–one and onto) and both f and f^{-1} are continuous. The notation $X \approx Y$ means that X is homeomorphic to Y.

Two topological spaces are, then, homeomorphic if there is a one–one correspondence between them as sets which also makes the open sets correspond. Homeomorphic spaces are considered as essentially the same. One of the main problems in topology is to find methods of deciding when two spaces are homeomorphic or not.

To describe a topological space it is not necessary to describe completely the open sets. This can often be done more simply using the notion of a "basis" for the topology:

2.8. Definition. If X is a topological space and **B** is a collection of subsets of X, then **B** is called a *basis* for the topology of X if the open sets are precisely the unions of members of **B**. (In particular, the members of **B** are open.) A collection **S** of subsets of X is called a *subbasis* for the topology of X if the set **B** of *finite* intersections of members of **S** is a basis.

Note that *any* collection **S** of subsets of any set X is a subbasis for *some* topology on X, namely, the topology for which the open sets are the arbitrary unions of the finite intersections of members of **S**. (The empty set and whole set X are taken care of by the convention that an intersection of an empty collection of sets is the whole set and the union of an empty collection of sets is the empty set.) Thus, to define a topology, it suffices to specify some collection of sets as a subbasis. The resulting topology is called the topology "generated" by this subbasis.

In a metric space the collection of ϵ-balls, for all $\epsilon > 0$, is a basis, So is the collection of ϵ-balls for $\epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots$.

Here are some examples of topological spaces:

1. (Trivial topology.) Any set X with only the empty set and the whole set X as open.
2. (Discrete topology.) Any set X with all subsets being open.
3. Any set X with open sets being those subsets of X whose complements are finite, together with the empty set. (That is, the closed sets are finite sets and X itself.)

4. $X = \omega \cup \{\omega\}$ with the open sets being all subsets of ω together with complements of finite sets. (Here, ω denotes the set of natural numbers.)
5. Let X be any partially ordered set. For $\alpha \in X$ consider the one-sided intervals $\{\beta \in X | \alpha < \beta\}$ and $\{\beta \in X | \alpha > \beta\}$. The "order topology" on X is the topology generated by these intervals. The "strong order topology" is the topology generated by these intervals together with the complements of finite sets.
6. Let $X = I \times I$ where I is the unit interval $[0, 1]$. Give this the "dictionary ordering," i.e., $(x, y) < (s, t) \Leftrightarrow$ either $x < s$ or ($x = s$ and $y < t$). Let X have the order topology for this ordering.
7. Let X be the real line but with the topology generated by the "half open intervals" $[x, y)$. This is called the "half open interval topology."
8. Let $X = \Omega \cup \{\Omega\}$ be the set of ordinal numbers up to and including the least uncountable ordinal Ω; see Theorem B.28. Give it the order topology.

2.9. Definition. A topological space is said to be *first countable* if each point has a countable neighborhood basis.

2.10. Definition. A topological space is said to be *second countable* if its topology has a countable basis.

Note that all metric spaces are first countable. Some metric spaces are not second countable, e.g., the space consisting of any uncountable set with the metric $\text{dist}(x, y) = 1$ if $x \neq y$, and $\text{dist}(x, x) = 0$ (which yields the discrete topology).

Euclidean spaces are second countable since the ϵ-balls, with ϵ rational, about the points with all rational coordinates, is easily seen to be a basis.

2.11. Definition. A sequence $f_1, f_2, \ldots$ of functions from a topological space X to a metric space Y is said to *converge uniformly* to a function $f: X \to Y$ if, for each $\epsilon > 0$, there is a number n such that $i > n \Rightarrow \text{dist}(f_i(x), f(x)) < \epsilon$ for all $x \in X$.

2.12. Theorem. *If a sequence $f_1, f_2, \ldots,$ of continuous functions from a topological space X to a metric space Y converges uniformly to a function $f: X \to Y$, then f is continuous.*

PROOF. Given $\epsilon > 0$, let n_0 be such that

$$n \geq n_0 \quad \Rightarrow \quad \text{dist}(f(x), f_n(x)) < \epsilon/3 \qquad \text{for all} \quad x \in X.$$

Given a point x_0, the continuity of f_{n_0} implies that there is a neighborhood N of x_0 such that $x \in N \Rightarrow \text{dist}(f_{n_0}(x), f_{n_0}(x_0)) < \epsilon/3$. Thus, for any $x \in N$ we have

$$\begin{aligned}\text{dist}(f(x), f(x_0)) &\leq \text{dist}(f(x), f_{n_0}(x)) + \text{dist}(f_{n_0}(x), f_{n_0}(x_0)) + \text{dist}(f_{n_0}(x_0), f(x_0)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$ □

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