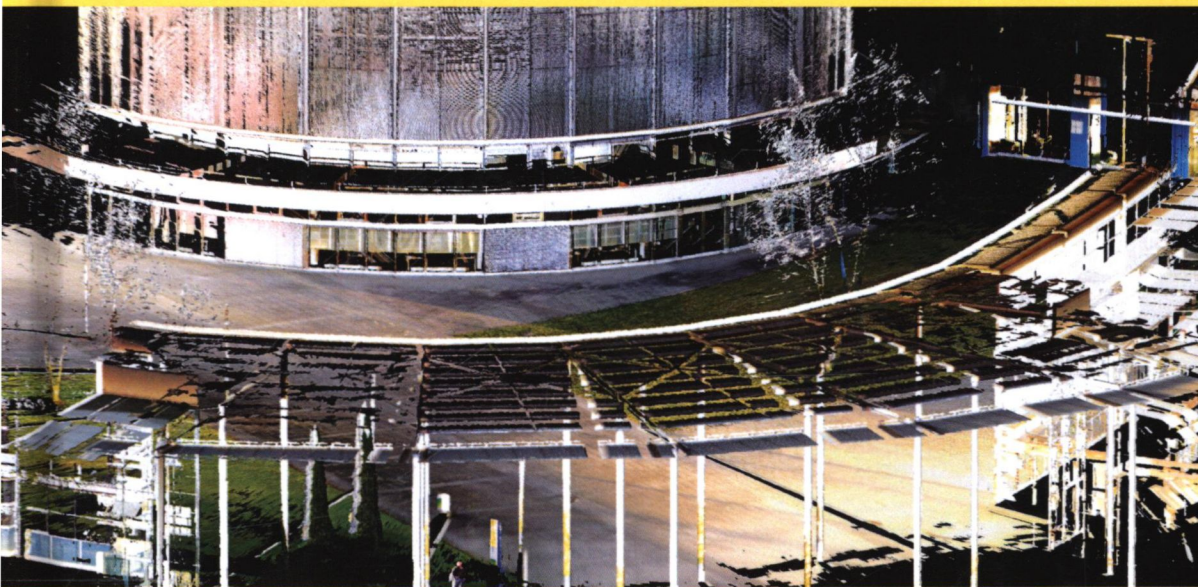


**NUMERICAL METHODS IN ENGINEERING SERIES**

**GEOMETRIC MODELING AND APPLICATIONS SET**



**Volume 1**

**Meshing, Geometric Modeling  
and Numerical Simulation 1**

*Form Functions,  
Triangulations and Geometric Modeling*

**Houman Borouchaki  
Paul Louis George**

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# GEOMETRIC MODELING AND APPLICATIONS SET

Coordinated by Marc Daniel

Triangulations, and more precisely meshes, are at the heart of many problems relating to a wide variety of scientific disciplines, and in particular numerical simulations of all kinds of physical phenomena. In numerical simulations, the functional spaces of approximation used to search for solutions are defined from meshes, and in this sense these meshes play a fundamental role. This strong link between the meshes and functional spaces leads us to consider advanced simulation methods in which the meshes are adapted to the behaviors of the underlying physical phenomena. This book presents the basic elements of this meshing vision.

These mesh adaptations are generally governed by a *posteriori* error estimators representing an increase of the error with respect to a size or metric. Independently of this metric of calculation, compliance with a geometry can also be calculated using a so-called geometric metric. The notion of mesh thus finds its meaning in the metric of its elements.

**Houman Borouchaki** is Professor at the University of Technology of Troyes (UTT) in France. He has been conducting research on mesh-related issues for more than 30 years, and is the author of several pieces of meshing software used in industry.

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# **Meshing, Geometric Modeling and Numerical Simulation 1**

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Meshing, Geometric Modeling and Numerical Simulation 1

*For Anahita, Amitisse and Hilda*

**Geometric Modeling and Applications Set**

coordinated by  
Marc Daniel

Volume 1

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## Foreword

The “Geometric Modeling and Applications” series is made up of five volumes of research on geometric modeling. The act of geometric modeling is an age-old one. To look at a few key points in its history: geometers in antiquity devoted themselves to it, though, of course, it had a different form from that which we know today; Descartes, in the mid-17th Century, worked on partitioning of planes; and Voronoï established concepts relating to this toward the end of the 19th Century.

But geometric modeling, as we understand it now, can be considered to be linked to the use of computers, especially since this tool has become ubiquitous. We are tempted to fix the scientific recognition of this activity to the date of the “First Conference on Computer Aided Geometric Design”, which took place in March 1974 at the University of Utah. This was organized by R. Barhnill and R. Riesenfeld, whose presentations may be found in the book “Computer Aided Geometric Design, Academic Press, 1974”, which clearly associates geometric modeling with the computer.

Since then, many detailed and high-quality books have been published in this field. One of the pioneering works was, undoubtedly, that authored by D. Faux and M. Pratt, “Computational Geometry for Design and Manufacture, Ellis Horwood Publishers, 1979”. Some books went through several editions, taking note of the advances made in research in the field. In this context, bringing out a new general book on the subject would mark no significant scientific progress.

Moreover, for any object or entity that is to be modeled, there is no single geometric model. Instead, there are several geometric models, each being adapted to the processes to be carried out. For example, the model to finely analyze the shape of an object is necessarily different from models that are used for mechanical computations on this same object. We have, therefore, chosen to focus on specific points of research in fields that we find particularly important. Thus, the five references given here, written by specialists, bring you up to date on advances in research in five domains:

- **“Constraints and geometric modeling”**, by *Dominique Michelucci, Pascal Schreck and Pascal Mathis*, looks at geometric problems that can be approached by describing the constraints, geometric or more complex, that a solution must verify, rather than directly giving the geometric definition of such a solution.

- **“Geometric modeling of fractal forms CAD”**, by *Christian Gentil, Dmitry Sokolov and Gilles Gouaty*, discusses the definition of fractal forms: this formalism makes it possible to reuse smooth forms of geometric models that are classic today to define forms that have not yet been explored till present.
- **“Meshing, geometric modeling and numerical simulation”**, in two volumes, by *Houman Borouchaki and Paul-Louis George*, details methodologies advanced for the construction of meshes and geometric modeling for numerical simulation with applications, especially in mechanics.
- **“Geometric and Topological Mesh Feature Extraction for 3D Shape Analysis”**, by *Jean-Luc Mari, Franck Hétroy-Wheeler and Gérard Subsol*, studies the analysis of three-dimensional-meshed shapes via the extraction of characteristics, both geometric and topological. The fields of application are also detailed to illustrate the transdisciplinary nature of this work.
- **“Relevant Triangulations and Samplings for 3D Shapes”**, by *Raphaëlle Chaine and Julie Digne*, discusses sampling, meshing and compression of free forms from a large quantity of real data from a virtual, interactive deformation process.

These are books that present recent research. The reader can find, in the general books on geometric modeling that were mentioned before, elements to fill any gaps they may have and to enable them to read the books in this series.

Happy reading.

Marc DANIEL

## Introduction

Triangulations and, more precisely, meshes, with the subtle difference that separates these two entities, lie at the heart of any number of problems that arise from varied scientific disciplines. A triangulation or a mesh is a discrete representation, using simple geometric elements (triangle, quadrilateral, tetrahedron, etc., any arbitrary polygon or polyhedron), of a domain that may be an object or a region of which we want a discrete, spatial description. There are, thus, many applications, including numerical simulations of any kind of physical problem, though not restricted to these. In particular, a discrete representation of a (volumetric) object or a surface may simply be seen as a geometric modeling problem as is. This book adopts a double point of view, as indicated by its title, and we will look both at the use of meshes in numerical simulations (the finite element method, especially) with, of course, the underlying constraints, as well as the use of these meshes for the (discrete) modeling of geometry.

The literature on triangulations and meshes may be classified in two chief categories: one more purely mathematical and geometric, the other more oriented toward industrial applications (numerical simulations) with, of course, though not always, relations between these categories.

The first point of view is covered by the computation geometry community, which studies (among others) Delaunay triangulations in all cuts, definitions, properties, construction algorithms and the complexity of these algorithms. They also study some applications of these triangulations. Nonetheless, relatively recently, mesh generation problems are also being studied, but under a more theoretical angle, generally relating to situations that allow for the use of Delaunay triangulations for which robust construction methods as well as interesting geometric properties have been known for a long time. This (somewhat monotheistic) philosophy necessarily imposes limits on the nature of the problems worked (workable). The first reference book on these subjects is that of Preparata and Shamos [Preparata, Shamos-1985] published in 1985. This was followed by several others, among which we cite two by Edelsbrunner [Edelsbrunner-1987] and [Edelsbrunner-2001], that of Yvinec and Boissonnat [Boissonnat, Yvinec-1997], by Dey [Dey-2007] and winding up the list with the book by Cheng *et al.* [Cheng *et al.* 2012], which was published in 2012. With a few exceptions, the orientation chosen by these references is not always guided by the preoccupations of mathematicians, engineers and the world of numerical simulation in general.



It is this very need for numerical simulations, in particular (and historically) in solid or fluid mechanics, that has led to the emergence of a “meshing” community among mathematicians and engineers. Without fear of contradiction, we can state that the very first book on meshes that was seen from this point of view is that of Thompson *et al.* [Thompson *et al.* 1985], which was also published in 1985. This book essentially discusses structured meshes or *grids*<sup>1</sup> while the first truly generalist book, discussing all types of meshes, structured or not, is that by George [George-1991], which dates back to 1991. Over the years and with the evolution of computers and methods as well as the increasingly complex nature of the meshes to consider, other books have been published. Let us mention books by George and Borouchaki [George, Borouchaki-1998], published in 1997, which revisits meshing methods based on Delaunay algorithms; by Carey [Carey-1997], also in 1997, which gives a very pedagogic presentation of methods; by Topping and co-authors [Topping *et al.* 2004], in 2004, cited in order to be exhaustive; by Frey and George [Frey, George-2008], in 2000, with a second edition in 2008, which returns to the general view and incorporates newly appeared techniques; by Löhner [Löhner-2008], in 2002, with a second edition that came out in 2008, which abounds in innovative details; up to the recent book by Lo [Lo-2015], published in 2015, which, among others, sheds new light on some problems. In this context, we may wonder what motivated the writing of this book. We will attempt to answer this question (and also try, by doing so, to arouse the reader’s curiosity).

The first observation is that the few books on this subject are already either a little old or rather classic in the way they discuss the approaches and the methods concerning the many questions linked to meshes. The second remark is that, evidently, new questions have appeared relatively recently, e.g. everything related to (finite) elements or patches of higher degrees, metrics and their links to interpolation errors, geometric modeling by meshes, numerical simulation via advanced methods (adaptation of meshes) and so on. This remark immediately calls forth another: we will resolutely adopt a double point of view by considering finite elements (or the constitutive elements of a mesh) as geometric patches and vice versa, thus establishing a link between the Lagrange world of finite elements and the Bézier world of CAD. This choice, as we will see, noticeably changes the manner in which we perceive the problems and, consequently, the manner in which we try to find solutions. In this spirit, this book in two volumes, beyond a set of subjects that we may qualify as classic (and, thus, essential), approaches many subjects that are very recent or even completely novel. Moreover, and this may be more surprising (and certainly rare), some methods (which we find in the literature) have been deliberately left out as they were deemed to not be of great interest and, *a contrario*, some methods are described in a manner that is, at the very least, nuanced and even critical. Indeed, it seemed pertinent to share our perception, even though this is subjective, up to a point, of these methods, perhaps to prevent the reader from going down paths that we think are dead ends. Having specified this, this book provides the necessary basis for a course on meshes and related problems at the masters level and it may serve as a technical reference for engineering students in science and, more generally, engineers using numerical simulations. We will give a brief description of the content of the chapters in both volumes.

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1. The grid-type meshes are generated by solving partial derivative equations, for example elliptical equations. It must be noted that other, more recent books on this theme, followed this pioneering work, but have not been mentioned here.

◇ Volume 1:

We first give a chapter-wise overview of the first volume of this book. In the first three chapters, this volume introduces the basic concepts related to finite elements seen through their shape functions either as finite elements or as patches. In addition to the classic expression via Lagrange polynomials (complete, reduced or rational) (Chapter 1), we give the equivalent formulation based on Bézier forms (Chapter 2), which makes it possible to easily find the conditions for the geometric validity of these finite elements (or patches) whether they are straight or curved and whatever their degree (Chapter 3). Triangulation problems are the subject of the next three chapters, where we specify vocabulary and the basic concepts that make it possible to describe the different construction methods for any triangulation (Chapter 4), Delaunay triangulation (Chapter 5) as well as the triangulation (of a domain) with constraints (Chapter 6). In the next two chapters, we use the concepts introduced earlier to discuss the vast problem of geometric modeling of a domain in its various aspects. Geometric modeling, from our point of views (meshes and numerical simulations), consists of construction of a discrete representation from a continuous representation and vice versa. Different methods are described in Chapters 7, while Chapter 8 gives several significant examples for the application of these methods. Chapter 9 brings together some basic algorithms and formulae related to finite elements (patches) and triangulations, to persuade the reader to go beyond a theoretical viewpoint, to a practical viewpoint, which we believe is essential.

◇ Volume 2:

We share the contents of Volume 2 [Borouchaki-2018] of this book in advance. This volume (finally) approaches mesh problems and begins with a detailed description of the concept of metric, a concept which will be seen to be fundamental in all that follows. The first two chapters, Chapters 1 and 2, thus focus on metrics. We introduce the concept of metric, we demonstrate their properties and their relations with the geometry of elements of a mesh and how to control interpolation errors during the resolution of a problem (by finite elements). The construction methods of meshes and their optimization are the focus of the following five chapters: Chapter 3 (mesh for a curve), Chapter 4 (simplicial meshes), Chapter 5 (non-simplicial meshes), Chapter 6 (meshes of higher degree) and Chapter 7 (optimizing meshes). We will then discuss the large subject of the adaptation of meshes, controlling the solutions via error estimates and corresponding interpolation metrics in Chapter 8. The use of a dose or a certain dose of parallelism (based on its multiple forms) is seen through Chapter 9. Chapter 10 illustrates, using concrete examples, a series of applications of the methods and methodologies introduced and described through the different chapters of both volumes of this book. As in Volume 1, the final chapter, Chapter 11, gives practical observations on some of the algorithms from Volume 2, especially how to use and manipulate metrics.

◇ Finally, at the end of this second volume, we sketch out future prospects.

\*

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