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Topics in Hyperplane Arrangements

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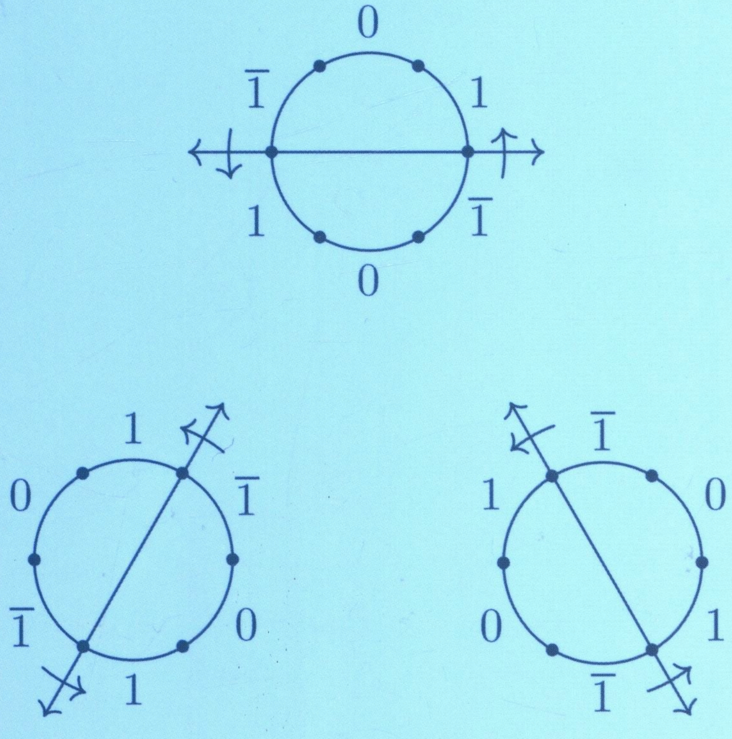
Swapneel Mahajan



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This monograph studies the interplay between various algebraic, geometric and combinatorial aspects of real hyperplane arrangements. It provides a careful, organized and unified treatment of several recent developments in the field, and brings forth many new ideas and results. It has two parts, each divided into eight chapters, and five appendices with background material.

Part I gives a detailed discussion on faces, flats, chambers, cones, gallery intervals, lunes and other geometric notions associated with arrangements. The Tits monoid plays a central role. Another important object is the category of lunes which generalizes the classical associative operad. Also discussed are the descent and lune identities, distance functions on chambers, and the combinatorics of the braid arrangement and related examples.

Part II studies the structure and representation theory of the Tits algebra of an arrangement. It gives a detailed analysis of idempotents and Peirce decompositions, and connects them to the classical theory of Eulerian idempotents. It introduces the space of Lie elements of an arrangement which generalizes the classical Lie operad. This space is the last nonzero power of the radical of the Tits algebra. It is also the socle of the left ideal of chambers and of the right ideal of Zie elements. Zie elements generalize the classical Lie idempotents. They include Dynkin elements associated to generic half-spaces which generalize the classical Dynkin idempotent. Another important object is the lune-incidence algebra which marks the beginning of noncommutative Möbius theory. These ideas are also brought upon the study of the Solomon descent algebra.

The monograph is written with clarity and in sufficient detail to make it accessible to graduate students. It can also serve as a useful reference to experts.

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Topics in Hyperplane Arrangements

Preface

Synopsis

The goal of this monograph is to study the interplay between various algebraic, geometric and combinatorial aspects of real hyperplane arrangements. The text contains many new ideas and results. It also gathers and organizes material from various sources in the literature, sometimes highlighting previously unnoticed connections. We briefly outline the contents below. They are explained in more detail in the main introduction.

We provide a detailed discussion on faces, flats, chambers, cones, gallery intervals, lunes, the support map, the case and base maps, and other geometric notions associated to real hyperplane arrangements. We show that any cone can be optimally decomposed into lunes. We introduce the category of lunes. This beautiful structure is intimately related to the substitution product of chambers a generalization of the classical associative operad). The classical case is obtained by specializing to the braid arrangement. We give several generalizations of the classical identity of Witt from Coxeter theory under the broad umbrella of descent and lune identities. The topological invariant involved here is the Euler characteristic of a relative pair of cell complexes. We generalize a well-known factorization theorem of Varchenko to cones, and also initiate an abstract approach to distance functions on chambers.

The main algebraic objects are the Birkhoff monoid and the Tits monoid, and their linearized algebras. The former is commutative and its elements are the flats of the arrangement, while the latter is not commutative and its elements are the faces. A module whose elements are chambers also plays a central role. Both monoids carry natural partial orders. The Birkhoff monoid is a lattice and its product is the join operation in the lattice. One may think of the Tits monoid as a noncommutative lattice. Its abelianization is the Birkhoff monoid, via the map that sends a face to the flat which supports it. We introduce the Janus monoid which is built out of the Tits and Birkhoff monoids.

We initiate a noncommutative Möbius theory of the Tits monoid and relate it to the representation theory of its linerization which is the Tits algebra. The central object is the lune-incidence algebra, which is a certain reduced incidence algebra of the poset of faces. It contains noncommutative zeta functions characterized by lune-additivity, and noncommutative Möbius functions characterized by the noncommutative Weisner formula. This theory lifts the usual Möbius theory for lattices, where the central object is the incidence algebra of the lattice of flats.

We introduce Lie and Zie elements. The latter belong to the Tits algebra, and the former to the module of chambers. The space of Zie elements is a right ideal of the Tits algebra. Any special Zie element defines an idempotent operator on chambers whose image is the space of Lie elements. To any generic half-space, we associate a special Zie element called the Dynkin element. Its action on chambers generalizes the left bracketing operator in classical Lie theory. We define a substitution product and establish a presentation of Lie. This generalizes the familiar presentation of the classical Lie operad. Antisymmetry is encoded in the notion of orientation of the rank-one arrangement and the Jacobi identity in the form of a linear relation among chambers obtained by "unbracketing" lines of the rank-two arrangements. This is same as saying that the space of Lie elements is isomorphic, up to orientation, to the top cohomology of the lattice of flats. This generalizes a celebrated theorem due to the combined work of Joyal, Klyachko and Stanley. We introduce the Lie-incidence algebra and show that it is isomorphic to the Tits algebra. This is intimately connected to the two-sided Peirce decomposition of the Tits algebra. The latter can be understood in terms of left and right Peirce decompositions of chambers and Zie elements respectively.

The Birkhoff algebra is split-semisimple. For the Tits algebra, complete systems of primitive orthogonal idempotents are in correspondence with algebra sections of the support map. We obtain many interesting characterizations of such sections. This aspect of the theory generalizes the classical theory of Eulerian idempotents. Noncommutative zeta and Möbius functions, and special Zie families are among the various concepts in correspondence. For reflection arrangements, there is a similar theory for the subalgebra of the Tits algebra invariant under the action of the Coxeter group. (The opposite of this algebra is the Solomon descent algebra.)

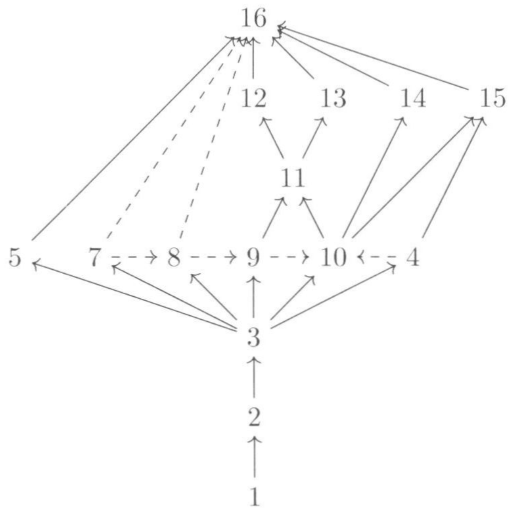
Precedents

This work benefits from and builds on some important recent developments. For the representation theory of the Tits algebra, we mention work of Brown, Diaconis and Saliola propelled by a landmark paper of Bidigare, Hanlon and Rockmore. (Older work of Solomon on the descent algebra has also been influential.) Some of these results are given in the generality of left regular bands and even bands. Further generalizations of this kind appear in work of Steinberg. For Lie theory, we mention work of Barcelo, Bergeron, Björner, Garsia, Patras, Reutenauer and Wachs. Saliola's work also implicitly contains elements of Lie theory. Explicit references to Lie are made only for the braid arrangement and the reflection arrangement of type B. The work of Joyal, Klyachko and Stanley relating Lie to order homology is for the braid arrangement. On the other hand, related results on order homology in the literature are usually given in the generality of arrangements or beyond. There have been several other contributors; most of them are mentioned in the main introduction. Two new entrants are the mathematicians Janus and Zie.

Organization

The text is organized in two parts. In Part I, the emphasis is on set-theoretic objects associated to hyperplane arrangements such as posets, monoids and the action of monoids on sets. In Part II, the emphasis is on linear objects such as algebras and their modules. There is a Notes section at the end of each chapter where detailed references to the literature, including discussions on alternative terminology and notation, are provided. Background information on topics such as Möbius functions, incidence algebras, representation theory of algebras and bands is provided in Appendices at the end of the main text. A notation index and a subject index are provided at the end of the book. Pictures and diagrams form an important component of our exposition which has a distinct geometric flavor. Numerous exercises are interspersed throughout the book.

The text is not meant to be read linearly from start to finish. We encourage readers to take up a particular chapter or section of their interest and backtrack as necessary. As an aid, the diagram of interdependence of chapters and appendices is displayed below.



A $\longrightarrow$ 1,2,7

B $\longrightarrow$ 1,2,3,4

C $\longrightarrow$ 1,15,16

D $\longrightarrow$ 9,11,12,13,16

E $\longrightarrow$ 1,8

A directed path from i to j indicates that some basic familiarity with Chapter i is necessary before proceeding to Chapter j. A dashed arrow from i to j means that the dependence of Chapter j on Chapter i is minimal, that is, restricted to some section or example.

Chapter 6 is not shown in the above diagram. It discusses the braid arrangement, the reflection arrangement of type B and other examples. They are employed frequently in later chapters for illustration.

Readership

We have strived to keep the text self-contained and with minimum prerequisites with the objective of making it accessible to advanced undergraduate and beginning graduate students. We hope it also serves as a useful reference on hyperplane arrangements to experts. The book touches upon several fields of mathematics such as representation theory of monoids and associative algebras, posets and their incidence algebras, lattice theory, random walks, invariant theory, discrete geometry, algebraic and geometric combinatorics, and algebraic Lie theory.

Scope

The theory of hyperplane arrangements has grown enormously in several different directions in the past two decades. The text is not meant to be a comprehensive survey of the entire theory. For instance, topics such as singularities, integral systems, hypergeometric functions and resonance varieties find no mention in the book. For these, one may look at [**15, 121, 138, 157, 177, 329, 413**] and references therein.

Future directions

Our constructions are all based on the choice of a real hyperplane arrangement. It is apparent, moreover, that a central role is played by the Tits monoid of faces of the arrangement. It is tempting to try to extend the theory to more general classes of monoids, particularly bands and left regular bands. We have kept our focus on arrangements, although such generalizations offer a promising line of research. We also mention the Janus monoid, the category of lunes and noncommutative Möbius functions as important objects worthy of further study. Our choice of topics has mainly been guided by applications to the theory of species, operads and Hopf algebras which we plan to develop in future work.

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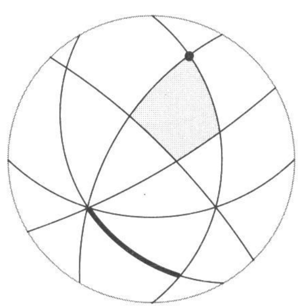


Introduction

Part I

Arrangements. (Chapter 1.) A hyperplane arrangement $\mathcal{A}$ is a set of hyperplanes (codimension-one subspaces) in a fixed real vector space. We assume that the number of hyperplanes is finite and all of them pass through the origin. The intersection of all hyperplanes is the central face. The rank of an arrangement is the dimension of the ambient vector space minus the dimension of the central face. An arrangement has rank 0 if it has no hyperplanes, rank 1 if it has one hyperplane, and rank 2 if it has at least two hyperplanes and all of them pass through a codimension-two subspace.

Flats and faces. (Chapter 1.) Subspaces obtained by intersecting hyperplanes are called the *flats* of the arrangement. We let $\Pi[\mathcal{A}]$ denote the set of flats. It is a graded lattice with partial order given by inclusion. The minimum element is the central face and the maximum element is the ambient space. The codimension-one flats are the hyperplanes. Each hyperplane divides the ambient space into two half-spaces. Their intersection is the given hyperplane. Subsets obtained by intersecting half-spaces, with at least one half-space chosen for each hyperplane, are called the *faces* of the arrangement. We let $\Sigma[\mathcal{A}]$ denote the set of faces. It is a graded poset under inclusion. The central face is the minimum element. However, there is no unique maximum face, so $\Sigma[\mathcal{A}]$ is *not* a lattice. A maximal face is called a *chamber*. We let $\Gamma[\mathcal{A}]$ denote the set of chambers. The linear span of any face is a flat. This defines a surjective map

$$\mathrm{s} : \Sigma[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}].$$

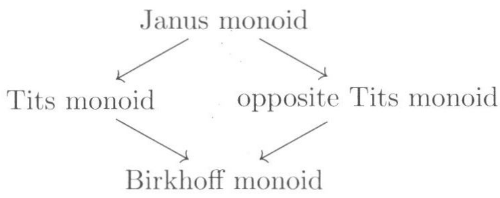
We call this the *support map*. It is order-preserving.

Birkhoff monoid and Tits monoid. (Chapter 1.) We view the lattice of flats $\Pi[\mathcal{A}]$ as a (commutative) monoid with product given by the join operation. We call this the *Birkhoff monoid*. For flats X and Y, their Birkhoff product is $\mathrm{X} \vee \mathrm{Y}$. The poset of faces $\Sigma[\mathcal{A}]$ is not a lattice. Nonetheless, it carries a (noncommutative) monoid structure. We call this the *Tits monoid*. It is an example of a left regular band (since it satisfies the axiom $xyx = xy$). For faces F and G, we denote their Tits product by FG. The set of chambers $\Gamma[\mathcal{A}]$ is a left $\Sigma[\mathcal{A}]$-set, that is, for F a face and C a chamber, FC is a chamber. The support map is a monoid homomorphism.

Janus monoid. (Chapter 1.) A *bi-face* is a pair (F, F') of faces such that F and F' have the same support. Let $\mathrm{J}[\mathcal{A}]$ denote the set of bi-faces. The operation

$$(F, F')(G, G') := (FG, G'F')$$

turns $\mathrm{J}[\mathcal{A}]$ into a monoid. We call it the *Janus monoid.* [1] It is the fiber product of the Tits monoid $\Sigma[\mathcal{A}]$ and its opposite $\Sigma[\mathcal{A}]^{\mathrm{op}}$ over the Birkhoff monoid $\Pi[\mathcal{A}]$. This can be pictured as follows.



The Janus monoid is a band (since every element is idempotent) which is neither left regular nor right regular in general.

Arrangements under and over a flat. (Chapter 1.) From a flat X of an arrangement $\mathcal{A}$, one may construct two new arrangements: $\mathcal{A}^{\mathrm{X}}$, the arrangement *under* X, and $\mathcal{A}_{\mathrm{X}}$, the arrangement *over* X. The former is the arrangement obtained by intersecting the hyperplanes in $\mathcal{A}$ with X, while the latter is the subarrangement consisting of those hyperplanes which contain X. For flats $\mathrm{X} \leq \mathrm{Y}$, the arrangement under Y in $\mathcal{A}_{\mathrm{X}}$ is the same as the arrangement over X in $\mathcal{A}^{\mathrm{Y}}$. We denote this arrangement by $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$.

Cones. (Chapter 2.) Subsets obtained by intersecting half-spaces (with no restriction) are called the *cones* of the arrangement. In particular, faces and flats are cones. (A hyperplane is the intersection of the two half-spaces it bounds.) Let $\Omega[\mathcal{A}]$ denote the set of all cones. It is a lattice under inclusion. The support map extends to an order-preserving map

$$\mathrm{c} : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}].$$

We call this the *case map.* It sends a cone to the smallest flat containing that cone. The case map is the left adjoint of the inclusion map $\Pi[\mathcal{A}] \to \Omega[\mathcal{A}]$. There is another order-preserving map

$$\mathrm{b} : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}]$$

which we call the *base map.* It sends a cone to the largest flat which is contained in that cone. The base map is the right adjoint of the inclusion map. Note that the base and case of a flat is the flat itself.

Cones whose case is the maximum flat are called top-cones. The poset of top-cones is a join-semilattice which is join-distributive, and in particular, graded and upper semimodular (Theorems 2.56, 2.58 and 2.60).

Lunes. (Chapters 3 and 4.) A cone is a *lune* if it has the property that for any hyperplane containing its base, the entire cone lies on one side of that hyperplane. Faces and flats are lunes. In general, any cone can be optimally cut up into lunes by using hyperplanes containing the base of the cone (Theorem 3.27). Finer decompositions can be obtained by using hyperplanes containing a fixed flat lying inside the base (Proposition 3.22). For instance, it is possible to cut a lune itself into smaller lunes. The optimal decomposition of a flat X is X itself (since it is a lune). An instance of a finer decomposition is to write X as a union of faces having support X.

[1] Janus Bifrons is a Roman god with two faces.

Lunes which are top-cones are called top-lunes. The poset of top-lunes under inclusion is graded (Theorem 4.9). We consider two partial orders on lunes. The first partial order is the inclusion of lune closures (and is the restriction of the partial order on cones), while the second is the inclusion of lune interiors. Both extend the partial order on top-lunes and are graded (Theorems 4.12 and 4.26).

Lunes can be composed when the case of the first lune equals the base of the second lune. This yields a category whose objects are flats and morphisms are lunes. We call it the *category of lunes*. It is internal to posets under the second partial order on lunes (Proposition 4.31). It also admits a nice presentation (Proposition 4.42). A lune with base X and case Y is the same as a chamber in the arrangement $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$. Using this, composition of lunes can be recast as follows. For any flat X, there is a map

$$\Gamma[\mathcal{A}^{\mathrm{X}}] \otimes \Gamma[\mathcal{A}_{\mathrm{X}}] \to \Gamma[\mathcal{A}].$$

We call this the substitution product of chambers, see (4.18).

Braid arrangement. (Chapters 5 and 6.) The braid arrangement is the motivating example for many of our considerations. The key observation is that for this arrangement, geometric notions of faces, flats, top-cones, and so on can be encoded by combinatorial notions of set compositions, set partitions, partial orders and so on. This correspondence between geometry and combinatorics is summarized in Table 6.2. The braid arrangement is an example of a reflection arrangement whose associated Coxeter group is the group of permutations. In the Coxeter case, one can define face-types and flat-types. Face-types are orbits of the set of faces under the Coxeter group action. Similarly, flat-types are orbits of the set of flats. For the braid arrangement, face-types and flat-types correspond to integer compositions and integer partitions.

Descent equation and lune equation. (Chapter 7.) Fix chambers C and D. The *descent equation* is $HC = D$. In other words, we need to solve for faces H such that the Tits product of H and C equals D. (This is related to descents of permutations in the case of the braid arrangement which motivates our terminology.) More generally, we can fix faces F and G, and consider the equation $HF = G$. In fact, one can do the following. For any left $\Sigma[\mathcal{A}]$-set h, the descent equation is $H \cdot x = y$, where x and y are fixed elements of h, the variable is H, and $\cdot$ denotes the action of $\Sigma[\mathcal{A}]$ on h. Apart from finding the solutions, there is also interest in computing the sum $\sum(-1)^{\mathrm{rk}(H)}$ as H ranges over the solution set, with $\mathrm{rk}(H)$ denoting the rank of H. For this, we attach to the solution set a relative pair (X, A) of cell complexes whose Euler characteristic is the given sum, see (7.32). By construction X is either a ball or sphere, but the topology of A is complicated in general. In our starting examples h is either $\Gamma[\mathcal{A}]$ or $\Sigma[\mathcal{A}]$. In these cases, A also has the topology of a ball or sphere. This leads to explicit identities, see (7.10) and (7.11a).

Fix a face H and a chamber D. The *lune equation* is $HC = D$. The difference is that now we need to solve for C. For a solution to exist H must be smaller than D. Assuming this condition, the solution set is precisely the set of chambers contained in some top-lune (which explains our terminology). More generally, an arbitrary lune can be obtained as the solution set of the equation $HF = G$ for some fixed H and G. Since lunes have the topology of a ball or sphere, we can again compute $\sum(-1)^{\mathrm{rk}(F)}$ explicitly, see (7.12a). An analysis with relative pairs, similar to the one for the descent equation, can be carried out for right $\Sigma[\mathcal{A}]$-sets h, see (7.41). The lune equation in this case is $x \cdot F = y$, with $x, y \in$ h.

Distance function and Varchenko matrix. (Chapter 8.) A hyperplane separates two chambers if they lie on its opposite sides. The distance between two chambers is defined to be the number of hyperplanes which separate them. Fix a scalar q, and define a bilinear form on the set of chambers $\Gamma[\mathcal{A}]$ by

$$\langle C, D\rangle := q^{\mathrm{dist}(C,D)}.$$

Here C and D are chambers and $\mathrm{dist}(C, D)$ denotes the distance between them. The determinant of this matrix factorizes with factors of the form $1-q^i$, see (8.41). In particular, the bilinear form is nondegenerate if q is not a root of unity.

More generally, assign a weight to each half-space, and define $\langle C, D\rangle$ to be the product of the weights of all half-spaces which contain C but do not contain D. Setting each weight to be q recovers the previous case. A factorization of the determinant of this matrix was obtained by Varchenko (Theorem 8.11). (He worked in the special case when the two opposite half-spaces bound by each hyperplane carry the same weight.) Lunes play a key role in the proof. The Varchenko matrix can be formally inverted using non-stuttering paths, see (8.30).

It is fruitful to consider a more general situation where we start with an arbitrary top-cone, and restrict the Varchenko matrix to chambers of this top-cone. The determinant of this matrix also factorizes. This more general result is given in Theorem 8.12. Specializing the top-cone to the ambient space recovers the previous situation. The special case of weights on hyperplanes is given in Theorem 8.22. This latter result has been obtained recently by Gente independent of our work.

Part II

Birkhoff algebra and Tits algebra. (Chapter 9.) The linearization of a monoid over a field $\Bbbk$ yields an algebra. Let $\mathsf{\Pi}[\mathcal{A}]$ denote the linearization of $\Pi[\mathcal{A}]$, and $\mathsf{\Sigma}[\mathcal{A}]$ denote the linearization of $\Sigma[\mathcal{A}]$ over $\Bbbk$. We call these the *Birkhoff algebra* and the *Tits algebra*, respectively. These are finite-dimensional $\Bbbk$-algebras (since the original monoids are finite). The linearization of $\Gamma[\mathcal{A}]$, denoted $\mathsf{\Gamma}[\mathcal{A}]$, is a left module over $\mathsf{\Sigma}[\mathcal{A}]$. One can linearize the support map as well to obtain an algebra homomorphism $\mathrm{s} : \mathsf{\Sigma}[\mathcal{A}] \twoheadrightarrow \mathsf{\Pi}[\mathcal{A}]$.

The Birkhoff algebra $\mathsf{\Pi}[\mathcal{A}]$ is isomorphic to $\Bbbk^n$, where n is the number of flats. In other words, $\mathsf{\Pi}[\mathcal{A}]$ is a split-semisimple commutative algebra (Theorem 9.2). (By a result of Solomon, this holds for any algebra obtained by linearizing a lattice.) The coordinate vectors of $\Bbbk^n$ yield a unique complete system of primitive orthogonal idempotents of $\mathsf{\Pi}[\mathcal{A}]$. We denote them by Q_{X}, as X varies over flats. The simple modules over $\mathsf{\Pi}[\mathcal{A}]$ are all one-dimensional, and given by $\mathsf{Q}_{\mathrm{X}} \cdot \mathsf{\Pi}[\mathcal{A}]$. Further, any module h is a direct sum of simple modules. More precisely, we have the Peirce decomposition [2]

$$\mathsf{h} = \bigoplus_{\mathrm{X}} \mathsf{Q}_{\mathrm{X}} \cdot \mathsf{h},$$

[2] A decomposition of a module using an orthogonal family of idempotents is called a Peirce decomposition.

and the simple module $\mathtt{Q}_X \cdot \sqcap[\mathcal{A}]$ occurs in the summand $\mathtt{Q}_X \cdot \mathsf{h}$ with multiplicity equal to its dimension (Theorems 9.7 and 9.8). As a consequence, the action of any element of $\sqcap[\mathcal{A}]$ on any module h is diagonalizable (Theorem 9.9).

The largest nilpotent ideal of an algebra A is called its radical, denoted $\mathrm{rad}(A)$. The Birkhoff algebra has no nonzero nilpotent elements, so $\mathrm{rad}(\sqcap[\mathcal{A}]) = 0$. In contrast, the Tits algebra has many nilpotent elements. In fact, $\mathrm{rad}(\Sigma[\mathcal{A}])$ is precisely the kernel of the (linearized) support map s, hence

$$\Sigma[\mathcal{A}]/\,\mathrm{rad}(\Sigma[\mathcal{A}]) \cong \sqcap[\mathcal{A}].$$

This was proved by Bidigare. We say that $\Sigma[\mathcal{A}]$ is an elementary algebra since the quotient by its radical is a split-semisimple commutative algebra. The simple modules over $\Sigma[\mathcal{A}]$ coincide with those over $\sqcap[\mathcal{A}]$ (since $\mathrm{rad}(\Sigma[\mathcal{A}])$ is forced to act by zero on such modules). However, a module of $\Sigma[\mathcal{A}]$ does not split as a direct sum of simple modules in general. (An example is provided by the module of chambers $\Gamma[\mathcal{A}]$.) Similarly, the action of an element of $\Sigma[\mathcal{A}]$ on a module h is not diagonalizable in general. Nonetheless, by taking a filtration of h, one can gain detailed information about the eigenvalues and multiplicities of the action (Theorem 9.44). This result for $\mathsf{h} := \Gamma[\mathcal{A}]$ was first obtained by Bidigare, Hanlon and Rockmore (Theorem 9.46); their motivation for considering this problem came from random walks. The above line of argument was given by Brown.

Any left module h over the Tits algebra has a *primitive part* which we denote by $\mathcal{P}(\mathsf{h})$. It consists of those elements of h which are annihilated by all faces except the central face (which acts by the identity). Dually, any right module h has a *decomposable part* which we denote by $\mathcal{D}(\mathsf{h})$. The duality is made precise in Proposition 9.61.

Janus algebra. (Chapter 9.) Let $\mathsf{J}[\mathcal{A}]$ denote the linearization of $\mathrm{J}[\mathcal{A}]$. We call this the *Janus algebra*. Just like the Tits algebra, the Janus algebra is elementary, and its split-semisimple quotient is the Birkhoff algebra. Interestingly, the Janus algebra admits a deformation by a scalar q. When q is not a root of unity, the q-Janus algebra is in fact split-semisimple, that is, isomorphic to a product of matrix algebras over $\Bbbk$. There is one matrix algebra for each flat X, with the size of the matrix being the number of faces with support X (Theorem 9.75). As a consequence, the q-Janus algebra, for q not a root of unity, is Morita equivalent to the Birkhoff algebra (Theorem 9.76). This is completely different from what happens for $q = 1$.

Eulerian idempotents. (Chapter 11.) Let us go back to the Tits algebra $\Sigma[\mathcal{A}]$. An *Eulerian family* $\mathtt{E}$ is a complete system of primitive orthogonal idempotents of $\Sigma[\mathcal{A}]$. Eulerian families are in correspondence with algebra sections $\sqcap[\mathcal{A}] \hookrightarrow \Sigma[\mathcal{A}]$ of the support map s. The construction of such sections is the idempotent lifting problem in ring theory. For elementary algebras, lifts always exist and any two lifts are conjugate by an invertible element in the algebra. For each X, we let $\mathtt{E}_X$ denote the image of $\mathtt{Q}_X$ under an algebra section, thus, $\mathrm{s}(\mathtt{E}_X) = \mathtt{Q}_X$. The $\mathtt{E}_X$ are called *Eulerian idempotents* and constitute the Eulerian family $\mathtt{E}$. Apart from being elementary, the Tits algebra is also the linearization of a left regular band. This allows for many interesting characterizations of Eulerian families (Theorems 11.20, 11.40 and 15.44). A highlight here is a construction of Saliola which produces an Eulerian family starting with a homogeneous section of the support map. (A homogeneous section is equivalent to an assignment of a scalar $\mathtt{u}^F$ to each face F such that for any flat X, the sum of $\mathtt{u}^F$ over all F with support X is 1.) This construction employs the *Saliola lemma* (Lemma 11.12), which is an important property of any Eulerian family. For a good reflection arrangement, we give cancelation-free formulas for the Eulerian idempotents arising from the uniform homogeneous section (Theorem 11.53).

Diagonalizability. (Chapter 12.) An element of an algebra is diagonalizable if it can be expressed as a linear combination of orthogonal idempotents. All elements of the Birkhoff algebra are diagonalizable. However, that is not true for the Tits algebra. For instance, no nonzero element of the radical of $\Sigma[\mathcal{A}]$ is diagonalizable. Following another method of Saliola, one can characterize diagonalizable elements using existence of eigensections (Corollary 12.15). Examples include nonnegative elements (Theorem 12.20) and separating elements (Theorem 12.17). The separating condition was introduced by Brown. For separating elements, there is a formula for the eigensection (arising from the Brown-Diaconis stationary distribution formula (12.6)), and a formula for the Eulerian idempotents due to Brown, see (12.12) and (12.13). Apart from these families, we also consider diagonalizability of specific elements such as the Takeuchi element (12.23) and the Fulman elements (12.38). For the braid arrangement, these include the Adams elements; their diagonalization is given in (12.49).

Lie elements and JKS. (Chapters 10 and 14.) Recall that the Tits algebra $\Sigma[\mathcal{A}]$ acts on the space of chambers $\Gamma[\mathcal{A}]$. We put

$$\mathsf{Lie}[\mathcal{A}] := \mathcal{P}(\Gamma[\mathcal{A}]),$$

the primitive part of $\Gamma[\mathcal{A}]$. This is the space of *Lie elements*. We refer to this description of $\mathsf{Lie}[\mathcal{A}]$ as the Friedrichs criterion. There are other characterizations of $\mathsf{Lie}[\mathcal{A}]$ such as the top-lune criterion and the descent criterion. In the case of the braid arrangement, $\mathsf{Lie}[\mathcal{A}]$ is the space of classical Lie elements (the multilinear part of the free Lie algebra). The top-lune criterion extends a classical result of Ree for the free Lie algebra, while the descent criterion extends a result of Garsia. The top-lune criterion says the following: A Lie element is an assignment of a scalar x^C to each chamber C such that the sum of these scalars in any top-lune (containing more than one chamber) is zero. In fact, by cutting a top-lune into smaller top-lunes, it suffices to restrict to top-lunes whose base is of rank 1. The dimension of $\mathsf{Lie}[\mathcal{A}]$ equals the absolute value of the Möbius number of $\mathcal{A}$. There are many ways to deduce this, see for instance (10.24) or (11.63). There are also many interesting bases for $\mathsf{Lie}[\mathcal{A}]$. We discuss the Dynkin basis (which depends on a generic half-space) and the Lyndon basis (which depends on a choice function).

For any flat X, there is a map

$$\mathsf{Lie}[\mathcal{A}^{\mathrm{X}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \to \mathsf{Lie}[\mathcal{A}].$$

We call this the substitution product of Lie, see (10.28). It is obtained by restricting the substitution product of chambers. All Lie elements of $\mathcal{A}$ can be generated by repeated substitutions starting with Lie elements of rank-one arrangements (which incorporate antisymmetry), subject to the Jacobi identities in rank-two arrangements (Theorem 14.41). Antisymmetry can be visualized as follows.

$$(\overset{1}{\bullet} \qquad \overset{\bar{1}}{\bullet}) + (\overset{\bar{1}}{\bullet} \qquad \overset{1}{\bullet}) = 0.$$