BEAMS ON ELASTIC FOUNDATION

THEORY WITH APPLICATIONS IN THE FIELDS OF CIVIL AND MECHANICAL ENGINEERING

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PREFACE

The subject of this book is the analysis of elastically supported beams. The elastic support is provided here by a load-bearing medium, referred to as the "foundation," distributed continuously along the length of the beams. Such conditions of support can be found in a large variety of technical problems. In some of these problems the identity of the *beam* and the *foundation* can be easily established, as in the case of actual foundation structures or in the case of the railroad track. In other problems, however, which constitute perhaps the most fruitful field of application of this theory, the concept of beam and foundation is more of an abstract nature. Such conditions we find in networks of beams and in thin-walled tubes, shells, and domes, where the elastic foundation for the beam part is supplied by the resilience of the adjoining portions of a continuous elastic structure. Apart from the diversity of technical applications, there is a considerable variation possible in the fundamental subject itself. The flexural rigidity of the beam or the elasticity of the foundation may be a variable quantity; the axis of the beam may be straight or curved or the character of the applied loading may be axial, transverse, or torsional, in addition to a combination of end conditions to which any of these beams may be subjected. On the whole, however, all these problems are closely related through an affinity in their mathematical formulation. This renders the entire subject matter eminently suitable for a comprehensive treatment, which is the aim of the present volume.

In the course of this work much help was derived from the numerous publications on the subject, including several monographs in German and Russian, to which references are made in the footnotes. In attempting to form a comprehensive unit of all this material it has been found that many questions of interest to research men and practicing engineers have not yet been answered. This made it necessary to develop new solutions, to work out new cases of loadings, etc., the result of which is that a sizable portion of the material contained in this volume is of a kind that has not been published before. Among these new developments we may mention in particular the use of end-conditioning forces for producing beams of finite length under any combination of loading, the reduction of the problem of axially symmetrical deformation of conical and spherical shells to that of bending of beams on elastic foundation, and the introduction of the concept of foundation layers representing partial continuity in the material of the foundation. In addition to these a large number of new formulas for specific cases of loading and end conditions were worked out, together with illustrative examples, which appear interspersed throughout the text. Though the problems discussed are chiefly in the field of statics, the solutions developed in this connection may also be employed in other fields of mathematical physics, particularly in vibrations and acoustics.

There are two basic types of elastic foundations. The first type is characterized by the fact that the pressure in the foundation is proportional at every point to the deflection occurring at that point and is independent of pressures or deflections produced elsewhere in the foundation. Such a correlation between pressures and deflections implies a lack of continuity in the supporting medium, just as if it were made up of rows of closely spaced but independent elastic springs. The second type of foundation is furnished by an elastic solid which, in contrast to the first one, represents the case of complete continuity in the supporting medium. Though the first type is mathematically simpler, one should not regard it, as some investigators do, as an approximation or an "elementary" solution for the elastic solid foundation, because it has its own physical characteristics and significance. Foundations of the first type have by far the wider field of application in physical sciences, and most of the problems mentioned above can be reduced to elastic supporting conditions falling under this classification. For this reason the larger part of the book, nine chapters out of the ten, is devoted to problems arising in connection with such an essentially discontinuous type of foundation, and only the last chapter deals with cases in which the supporting body is an elastic continuum. Problems of continuity are introduced in the tenth chapter with a discussion of foundation layers which, with their varying and adjustable degree of continuity, form a useful transition between the two basic types of foundations mentioned above.

In the mathematical notation of this text a minor departure was made from existing practice in that capital initial letters are used in the otherwise customary notations for hyperbolic functions. The need for this arose from the fact that, owing to the nature of the subject, solutions often appeared in lengthy and sometimes perplexing combinations of trigonometric and hyperbolic functions. Thus it became highly desirable to accentuate the difference in notation between these two types of functions, and the use of a capital initial for the latter type was found to be a simple and effective way to achieve the purpose.

The first manuscript for this book was prepared in 1936–37 during the tenure of a Horace H. Rackham Postdoctorate Fellowship at the University of Michigan. Since then the material has been revised several times and enlarged until it has assumed its present form. The author takes this opportunity to express his deep appreciation and gratitude to the University of Michigan for granting the generous fellowship which made this undertaking possible and for supporting the publication of the ensuing results. During the work much encouragement and benefit were derived from personal contacts with Professor Stephen P. Timoshenko, who first aroused the author's interest in this subject and who proved to be a constant source of inspiration. It is also a pleasure to acknowledge the valuable assistance received from Professor Edward L. Eriksen, Dr. Merhyle F. Spotts, and Dr. Stewart Way. The author is greatly indebted to Dr. Eugene S. McCartney, editor of the University of Michigan Press, for his care in steering the publication through the press under wartime conditions and for the many constructive suggestions that both he and Miss Grace Potter, former assistant editor, have contributed.

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CHAPTER I

GENERAL SOLUTION OF THE ELASTIC LINE

In the major part of this work the analysis of bending of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. This assumption was introduced first by E. Winkler* in 1867 and formed the basis of H. Zimmermann's classical work† on the analysis of the railroad track, published in 1888. Though the early investigators thought chiefly of soil as the supporting medium, it was later found that there are other fields where the conditions of Winkler's assumption are much more rigorously satisfied. Two such fields of application were discovered to be of particular importance, and they are discussed in detail in the course of this book. One of these is concerned with networks of beams, which are characteristic in the construction of floor systems for ships, buildings, and bridges; the other deals with thin shells of revolution and includes such subjects as pressure vessels, boilers, and containers, as well as large-span modern reinforced concrete halls and domes. While the theory of beams on elastic foundation holds rigidly for most of the problems mentioned above, its application to soil foundations should be regarded only as a practical approximation. The physical properties of soils are obviously of a much more complicated nature than that which could be accurately represented by such a simple mathematical relationship as the one assumed by Winkler. There are, however, some important points which can be brought up in supporting the application of this theory to soil foundations. Under certain conditions the elasticity of soil is undeniable; it can propagate sound waves, for instance. Also, the second, and most debated, part of Winkler's assumption, that the foundation deforms only along the portion directly under loading, has, since A. Föppl's classical experiment,‡ often been found to be true for a large variety of soils. If we take these things into consideration, there is reason to believe that the Winkler theory, in spite of its simplicity, may often more accurately represent the actual conditions existing in soil foundations than do some of the more complicated analyses advanced in recent years and discussed in the last chapter of this book, where the foundation is regarded as a continuous isotropic elastic body. Which one of these theories to apply, and how much continuity in the supporting medium to assume, can be decided, however, in a given case only by physical testing of the material of the foundation under consideration.

* *Die Lehre von der Elastizität und Festigkeit* (Prag, 1867), p. 182.

† *Die Berechnung des Eisenbahnoberbaues* (Berlin, 1888; 2d ed., Berlin, 1930).

‡ A. Föppl, *Vorlesungen über technische Mechanik* (9th ed.; Leipzig, 1922), III, 258.

1. *The Differential Equation of the Elastic Line*

Consider a straight beam supported along its entire length by an elastic medium and subjected to vertical forces acting in the principal plane of the symmetrical cross section (Fig. 1). Because of this action the beam will deflect, producing continuously distributed reaction forces in the supporting medium. Regarding these reaction forces we make the fundamental assumption that their intensity p at any point is proportional to the deflection of the beam y at that point: $p = ky$ The reaction forces will be assumed to be acting vertically and opposing the deflection of the beam. Hence where the deflection is directed downward (positive) there will be a compression in the supporting medium, but, on the other hand, where the deflection happens to be negative, tension will be produced; for the present we suppose the supporting medium to be able to take up such tensile forces.

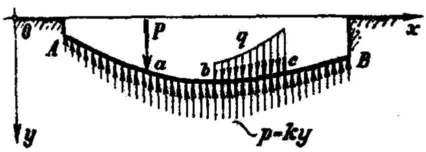

FIG. 1

The assumption $p = ky$ implies the statement that the supporting medium is elastic; in other words, that its material follows Hooke's law. Its elasticity, therefore, can be characterized by the force which, distributed over a unit area, will cause a deflection equal to unity. This constant of the supporting medium, k_0 lbs./in.^{3}, is called the *modulus of the foundation.*

Assume that the beam under consideration has a uniform cross section and that b is its constant width, which is supported on the foundation. A unit deflection of this beam will cause reaction bk_0 in the foundation; consequently, at a point where the deflection is y the intensity of distributed reaction (per unit length of the beam) will be

$$p \text{ lbs./in.} = bk_0 y. \tag{a}$$

For the sake of brevity we shall use the symbol k lbs./in.^{2} for b in. $\times$ k_0 lbs./in.^{3} in the following derivations, but it is to be remembered that this k includes the effect of the width of the beam and will be numerically equal to k_0 only if we deal with a beam of unit width.

While the loaded beam deflects, it is possible that besides the vertical reactions there may also be some horizontal (frictional) forces originating along the surface where the beam is in contact with the foundation. The influence of such horizontal forces on the deflection line will be shown in a later chapter; for the present their (possibly small) effect will not be considered, and the reaction forces on the foundation will be assumed to be vertical at every cross section.

Let us take an infinitely small element enclosed between two vertical* cross

* By this we assume that the slope is so small compared to unity that cross sections (normal to the elastic line) can be replaced by vertical sections. Such approximation cannot be used when investigating the effect of axial forces on the deflection of the beam (Chapter VI).

sections a distance dx apart on the beam under consideration. Assume that this element was taken from a portion where the beam was acted upon by a distributed loading q lbs./in. The forces exerted on such an element are shown in Figure 2. The upward-acting shearing force, Q, to the left of the cross section is considered positive, as is the corresponding bending moment, M, which is a clockwise moment acting from the left on the element (the moment of a positive Q). These positive directions for Q and M will be kept in all later derivations. Considering the equilibrium of the element in Figure 2, we find that the summation of the vertical forces gives

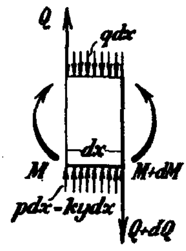

FIG. 2

$$Q - (Q + dQ) + ky\,dx - q\,dx = 0,$$

whence

$$\frac{dQ}{dx} = ky - q. \tag{b}$$

Making use of the relation $Q = dM/dx$, we can write

$$\frac{dQ}{dx} = \frac{d^2M}{dx^2} = ky - q. \tag{c}$$

Using now the known differential equation of a beam in bending, $EI(d^2y/dx^2) = -M$, and differentiating it twice, we obtain

$$EI\frac{d^4y}{dx^4} = -\frac{d^2M}{dx^2}. \tag{d}$$

Hence by using (c) we find

$$EI\frac{d^4y}{dx^4} = -ky + q. \tag{1}$$

This is the differential equation for the deflection curve of a beam supported on an elastic foundation. Along the unloaded parts of the beam, where no distributed load is acting, $q = 0$, and the equation above will take the form

$$EI\frac{d^4y}{dx^4} = -ky. \tag{2}$$

It will be sufficient to consider below only the general solution of (2), from which solutions will be obtained also for cases implied in (1) by adding to it a particular integral corresponding to q in (1).

Substituting $y = e^{mx}$ in (2), we obtain the characteristic equation

$$m^4 = -\frac{k}{EI},$$

which has the roots

$$m_1 = -m_3 = \sqrt[4]{\frac{k}{4EI}}\,(1 + i) = \lambda(1 + i),$$

$$m_2 = -m_4 = \sqrt[4]{\frac{k}{4EI}}\,(-1 + i) = \lambda(-1 + i).$$

The general solution of (2) takes the form

$$y = A_1e^{m_1x} + A_2e^{m_2x} + A_3e^{m_3x} + A_4e^{m_4x}, \tag{e}$$

where

$$\lambda = \sqrt[4]{\frac{k}{4EI}}. \tag{f}$$

Using

$$e^{i\lambda x} = \cos \lambda x + i \sin \lambda x,$$

$$e^{-i\lambda x} = \cos \lambda x - i \sin \lambda x,$$

and introducing the new constants C_1, C_2, C_3, and C_4, where

$$(A_1 + A_4) = C_1, \qquad i(A_1 - A_4) = C_2,$$

$$(A_2 + A_3) = C_3, \qquad i(A_2 - A_3) = C_4,$$

we can write (e) in a more convenient form:

$$y = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x). \tag{3a}$$

Here λ includes the flexural rigidity of the beam as well as the elasticity of the supporting medium, and is an important factor influencing the shape of the elastic line. For this reason the factor λ is called the *characteristic* of the system, and, since its dimension is length^{-1}, the term $1/\lambda$ is frequently referred to as the *characteristic length*. Consequently, λx will be an absolute number.

Expression (3a) represents the general solution for the deflection line of a straight prismatic bar supported on an elastic foundation and subjected to transverse bending forces, but with no q loading. An additional term is necessary where there is a distributed load. By differentiation of (3a) we get

$$\left.\begin{aligned}
\frac{1}{\lambda}\frac{dy}{dx} &= e^{\lambda x}[C_1(\cos\lambda x - \sin\lambda x) + C_2(\cos\lambda x + \sin\lambda x)] \\
&\qquad - e^{-\lambda x}[C_3(\cos\lambda x + \sin\lambda x) - C_4(\cos\lambda x - \sin\lambda x)], \\
\frac{1}{2\lambda^2}\frac{d^2y}{dx^2} &= -e^{\lambda x}(C_1\sin\lambda x - C_2\cos\lambda x) + e^{-\lambda x}(C_3\sin\lambda x - C_4\cos\lambda x) \\
\frac{1}{2\lambda^3}\frac{d^3y}{dx^3} &= -e^{\lambda x}[C_1(\cos\lambda x + \sin\lambda x) - C_2(\cos\lambda x - \sin\lambda x)] \\
&\qquad + e^{-\lambda x}[C_3(\cos\lambda x - \sin\lambda x) + C_4(\cos\lambda x + \sin\lambda x)].
\end{aligned}\right\} \quad (3\,\text{b–d})$$

Knowing that

$$\frac{dy}{dx} = \tan\theta, \quad -EI\frac{d^2y}{dx^2} = M, \quad \text{and} \quad -EI\frac{d^3y}{dx^3} = Q, \tag{g}$$

we can obtain the general expressions for the slope θ* of the deflection line as well as for the bending moment M and the shearing force Q from (3 b–d). The intensity of pressure in the foundation will be found from (3a) to be $p = ky$.

In applying these general equations, or corresponding ones including the term dependent on q, to particular cases the next step is to determine the constants of integration C_1, C_2, C_3, and C_4. These integration constants depend on the manner in which the beam is subjected to the loading and have constant values along each portion of the beam within which the elastic line and all its derivatives are continuous. Their values can be obtained from the conditions existing at the two ends of such continuous portions. Out of the four quantities (y, θ, M, and Q) characterizing the condition of an end, two are usually known at each end, from which sufficient data are furnished for the determination of the constants C.

When a beam is subjected to various loads the elastic line must be resolved into continuous portions (for example, A–a, a–b, b–c, and c–B in Fig. 1); then at the intermediate points the consideration of the material continuity of the beam will furnish the data for determining the integration constants for each of these portions.

Although from the point of view of mathematics the problem can be completely solved in this way,† the procedure is laborious and not well fitted to practical computation. The work can be considerably simplified, however, if the general solution is written in such a form that the integration constants obtain a physical interpretation in terms of the end conditions. This method of solution will be discussed in the next section.

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* On the basis of the approximate bending formula used above in the derivation (d) it is permissible to put $\tan\theta = \theta$.

† This method was used by K. Hayashi in his book *Theorie des Trägers auf elastischer Unterlage und ihre Anwendung auf den Tiefbau* (Berlin, 1921).

2. *Interpretation of the Integration Constants*

Assume a beam subjected to various loading (such as moment M, force P, and distributed load q) and take the origin of an x, y coordinate system at the left end of the beam (Fig. 3).

In (3 a–d) general expressions were obtained for the y, θ, M, and Q quantities of a beam in bending. Taking in these equations $x = 0$, we get the conditions at the left end of our beam as

$$\left.\begin{aligned} [y]_{x=0} &= y_0 = C_1 + C_3\,, \\ \left[\frac{dy}{dx}\right]_{x=0} &= \theta_0 = \lambda(C_1 + C_2 - C_3 + C_4), \\ \left[-EI\frac{d^2y}{dx^2}\right]_{x=0} &= M_0 = 2\lambda^2 EI(-C_2 + C_4), \\ \left[-EI\frac{d^3y}{dx^3}\right]_{x=0} &= Q_0 = 2\lambda^3 EI(C_1 - C_2 - C_3 - C_4). \end{aligned}\right\} \tag{a}$$

Expressing the C's as unknowns, we have, from the equations above,

$$\left.\begin{aligned} C_1 &= \tfrac{1}{2}y_0 + \frac{1}{4\lambda}\theta_0 + \frac{1}{8\lambda^3 EI}Q_0\,, \\ C_2 &= \frac{1}{4\lambda}\theta_0 - \frac{1}{4\lambda^2 EI}M_0 - \frac{1}{8\lambda^3 EI}Q_0\,, \\ C_3 &= \tfrac{1}{2}y_0 - \frac{1}{4\lambda}\theta_0 - \frac{1}{8\lambda^3 EI}Q_0\,, \\ C_4 &= \frac{1}{4\lambda}\theta_0 + \frac{1}{4\lambda^2 EI}M_0 - \frac{1}{8\lambda^3 EI}Q_0\,. \end{aligned}\right\} \tag{b}$$

Substituting these expressions for the C's in (3a) and putting $\frac{1}{2}(e^{\lambda x} + e^{-\lambda x}) = \text{Cosh}\,\lambda x$ and $\frac{1}{2}(e^{\lambda x} - e^{-\lambda x}) = \text{Sinh}\,\lambda x$, we find that the general equation of the elastic line will take the form

$$y_x = y_0 F_1(\lambda x) + \frac{1}{\lambda}\theta_0 F_2(\lambda x) - \frac{1}{\lambda^2 EI}M_0 F_3(\lambda x) - \frac{1}{\lambda^3 EI}Q_0 F_4(\lambda x)\cdots, \tag{c}$$

where

$$\begin{aligned} F_1(\lambda x) &= \text{Cosh}\,\lambda x\cos\lambda x, \\ F_2(\lambda x) &= \tfrac{1}{2}(\text{Cosh}\,\lambda x\sin\lambda x + \text{Sinh}\,\lambda x\cos\lambda x), \\ F_3(\lambda x) &= \tfrac{1}{2}\text{Sinh}\,\lambda x\sin\lambda x, \\ F_4(\lambda x) &= \tfrac{1}{4}(\text{Cosh}\,\lambda x\sin\lambda x - \text{Sinh}\,\lambda x\cos\lambda x). \end{aligned}$$

It is seen that in (c) the general solution was put in a form in which the previous integration constants were replaced by the y_0, θ_0, M_0, and Q_0 quantities existing at the end $x = 0$ of the beam. On account of this feature the method developed on the basis of (c) is termed the *method of initial conditions*;* because the simple interpretation of the integration constants it has a considerable advantage over the method outlined in the previous section.

A more generalized form of (c) can be obtained through the following reasoning: Assume that the y_0, θ_0, M_0, and Q_0 quantities are known; then we can proceed from the left end of the beam toward the right along the unloaded portion A–a until we arrive at the point where the first load is applied to the beam. Assume that the first loading is a concentrated moment M, as shown in Figure 3. Evidently this moment M must have an effect to the right ($x > u_M$) of its point of application similar to that which the initial moment M_0 had on the A–a portion ($0 < x < u_M$) of the elastic line. Seeing from (c) that the factor of M_0 was $-(1/\lambda^2EI)F_3(\lambda x)$, we can conclude that the moment M at a should have a modifying effect of $-(1/\lambda^2EI)F_3[\lambda(x - u_M)]M$ on the elastic line to the right of point a, where $x > u_M$. Consequently, we obtain the deflection curve on the portion a–b by adding this last expression to (c).

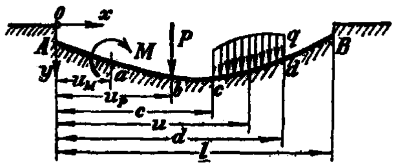

FIG. 3

In a similar way we find that the force P will have an influence $(1/\lambda^3EI)F_4[\lambda(x - u_P)]P$† on the deflection line to the right of point b. Finally, since the distributed loading q can be regarded as consisting of infinitesimal concentrated forces, we can conclude that its effect for the $x > c$ portion must be $(1/\lambda^3EI)\int_c^x qF_4[\lambda(x - u)]\,du$. For $x > d$ the upper limit of the integral becomes d. Summing up these results, we find the equation of the deflection line for such a case as that shown in Figure 3‡ to be

$$y_x = y_0F_1(\lambda x) + \frac{1}{\lambda}\theta_0F_2(\lambda x) - \frac{1}{\lambda^2EI}M_0F_3(\lambda x) - \frac{1}{\lambda^3EI}Q_0F_4(\lambda x)$$
$$- \frac{1}{\lambda^2EI}MF_3[\lambda(x - u_M)] + \frac{1}{\lambda^3EI}PF_4[\lambda(x - u_P)]$$
$$+ \frac{1}{\lambda^3EI}\int_c^x qF_4[\lambda(x - u)]\,du. \qquad (4a)$$

* This method was developed largely in Russia. See A. A. Umansky, *Analysis of Beams on Elastic Foundation*, Central Research Institute of Auto-Transportation (Leningrad, 1933); and *idem*, *Special Course in Structural Mechanics*, General Redaction of Literature of Building (Leningrad-Moscow, 1935), Part I. These publications contain also bibliographies of earlier Russian works.

† Here the sign of the term taken from (c) had to be changed, since the downward-acting force P represents a negative shear for the portion to the right of point b.

‡ The expression for the deflection line could be generalized in a still larger sense by including among the loadings concentrated changes in the deflection ordinates and in the slopes, and also by regarding distributed moments as a loading type. Expressions for such cases are to be found in Umansky, *opera cit.*

This equation includes the effect of M, P, and q acting on the beam between the left end ($x = 0$) and the point under consideration ($x = x$). If any of these loadings are absent on this portion of the beam the corresponding term in (4a) should be disregarded. By taking the consecutive derivatives of the equation above and putting

$$\frac{dF_1}{dx} = -4\lambda F_4, \qquad \frac{dF_2}{dx} = \lambda F_1, \qquad \frac{dF_3}{dx} = \lambda F_2, \quad \text{and} \quad \frac{dF_4}{dx} = \lambda F_3,$$

we obtain the expressions below for slope, moment, and shearing force:

$$\left.\begin{aligned}
\theta_x &= \theta_0 F_1(\lambda x) - \frac{1}{\lambda EI} M_0 F_2(\lambda x) - \frac{1}{\lambda^2 EI} Q_0 F_3(\lambda x) - 4\lambda y_0 F_4(\lambda x) \\
&\quad - \frac{1}{\lambda EI} M F_2[\lambda(x - u_M)] + \frac{1}{\lambda^2 EI} P F_3[\lambda(x - u_P)] \\
&\quad + \frac{1}{\lambda^2 EI} \int_c^x q F_3[\lambda(x - u)]\, du, \\
M_x &= M_0 F_1(\lambda x) + \frac{1}{\lambda} Q_0 F_2(\lambda x) + \frac{k}{\lambda^2} y_0 F_3(\lambda x) + \frac{k}{\lambda^3} \theta_0 F_4(\lambda x) \\
&\quad + M F_1[\lambda(x - u_M)] - \frac{1}{\lambda} P F_2[\lambda(x - u_P)] - \frac{1}{\lambda} \int_c^x q F_2[\lambda(x - u)]\, du, \\
Q_x &= Q_0 F_1(\lambda x) + \frac{k}{\lambda} y_0 F_2(\lambda x) + \frac{k}{\lambda^2} \theta_0 F_3(\lambda x) - 4\lambda M_0 F_4(\lambda x) \\
&\quad - 4\lambda M F_4[\lambda(x - u_M)] - P F_1[\lambda(x - u_P)] - \int_c^x q F_1[\lambda(x - u)]\, du.
\end{aligned}\right\} \quad (4\,\text{b–d})$$

It is seen that the initial conditions appear in equations (4 a–d) according to a systematic scheme. In each of these equations all the four initial conditions are present and the order of their succession is shifted by one place at a time as we proceed from (4a) to (4d). The same systematic shifting can be observed also in the F functions connected with the loading terms M, P, and q.

Putting $x = l$ into (4 a–d), we obtain the y_l, θ_l, M_l, and Q_l quantities for the right end of the beam as expressed in terms of the initial conditions and the loadings. These relations can then be used to determine the unknown initial conditions. As we have said, out of the four quantities which define the condition of one end of a beam, two are usually known at each end in every case. There remain two unknowns at each end; altogether there are four unknown quantities which can be determined from (4 a–d).

Consider for instance the beam in Figure 3 with both ends free. Here we have $M_0 = 0$, $Q_0 = 0$ and $M_l = 0$, $Q_l = 0$. Substituting these values in (4 a–d) we find that the left-hand side of (4c) and (4d) will be zero, while the right-hand side will contain only two unknown initial conditions, y_0 and θ_0. From the two simultaneous equations the two unknown quantities can be determined; then, substituting these, in turn, in the general expressions (4 a–d), we can proceed to calculate the y_x, θ_x, M_x, and Q_x values for any intermediate point on the beam. The outstanding feature of this method is the simple physical interpretation of the integration constants and the systematic order in which these constants appear in the equations. For practical computation, however, the method can be considered only if there are numerical tables of the F functions available, and even then more complicated loadings involve lengthy and intricate calculations.

3. *Method of Superposition*

In the preceding sections two different methods have been presented, both of them aiming to determine the integration constants from the prescribed end conditions of the elastic line. It has been seen that the main difficulty in applying the general solution to particular problems arises in the determination of the integration constants, which involves a considerable amount of work in both methods discussed.

These difficulties can be largely avoided by using the *method of superposition*.* The advantage of this method lies in the fact that the determination of the integration constants for a beam of unlimited length (an infinitely long beam) is very simple and that, consequently, the equation of the deflection line for any loading on the infinitely long beam can be obtained in a concise form. Such deflection formulas will be derived in Chapter II; in Chapter III it will be shown that by superposing the formulas obtained for the infinitely long beam solutions can be derived for beams of any length and with any loading and end conditions. This procedure will prove to be the simplest in the application to particular problems; it can be used also when, in addition to the lateral loads, axial forces or twisting moments are acting on the beam.†

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* The application of the method of superposition in the solution of beams of finite length on an elastic foundation was first proposed by the writer in a paper called "Analysis of Bars on Elastic Foundation," *Final Report of the Second International Congress for Bridge and Structural Engineering* (Berlin-Munich, 1936).

† The scheme in the method of initial conditions loses its periodical character when axial forces, in addition to the transverse loading, are acting on the beam.

CHAPTER II

BEAMS OF UNLIMITED LENGTH

I. The Infinite Beam

4. *Concentrated Loading*

Consider a beam of unlimited length in both directions (an infinite beam) subjected to a single concentrated force P at point O (Fig. 4). Because of the apparent symmetry of the deflection curve we need to consider only the half which is to the right of point O, the origin of the x, y rectangular coordinate system.

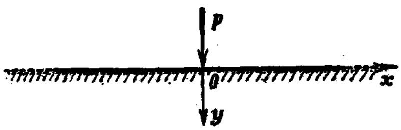

FIG. 4

In §1 we found that the general solution for the deflection curve of a beam subjected to transverse loading can be written as equation (3a):

$$y = e^{\lambda x}(C_1 \cos \lambda x + C_2 \sin \lambda x) + e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x). \tag{a}$$

In the present problem, dealing with a beam of unlimited length, it is reasonable to assume that in an infinite distance from the application of the load the deflection of the beam must approach zero, that is, if $x \to \infty$, then $y \to 0$. This condition can be fulfilled only if in the equation above the terms connected with $e^{\lambda x}$ vanish, which necessitates that in the case under discussion $C_1 = 0$ and $C_2 = 0$. Hence the deflection curve for the right part ($x > 0$) of the beam will take the form

$$y = e^{-\lambda x}(C_3 \cos \lambda x + C_4 \sin \lambda x). \tag{b}$$

From the condition of symmetry we know that

$$\left[\frac{dy}{dx}\right]_{x=0} = 0,$$

that is, $-(C_3 - C_4) = 0$, from which we find $C_3 = C_4 = C$. This last constant of the equation

$$y = Ce^{-\lambda x}(\cos \lambda x + \sin \lambda x) \tag{c}$$

can be obtained from the consideration that the sum of the reaction forces will keep equilibrium with the load P, that is,

$$2\int_0^\infty ky\,dx = P.$$

Since $2kC \int_0^\infty e^{-\lambda x}(\cos \lambda x + \sin \lambda x)\, dx = 2kC(1/\lambda)$, from $2kC(1/\lambda) = P$ we get $C = P\lambda/2k$, and, substituting this in (c) above, we have

$$y = \frac{P\lambda}{2k} e^{-\lambda x} (\cos \lambda x + \sin \lambda x), \tag{d}$$

which gives the deflection curve for the right side ($x \geqq 0$) of the beam. This deflection curve is a wavy line with decreasing amplitude (Fig. 5a). The deflection under the load is $y_0 = P\lambda/2k$; the zero points of the line are where $\cos \lambda x + \sin \lambda x = 0$, that is, at the consecutive values of $\lambda x = \frac{3}{4}\pi, \frac{7}{4}\pi, \frac{11}{4}\pi$, etc.

Taking the successive derivatives of y (see [d]) with respect to x, we obtain the expressions for θ, M, and Q on the right side of the beam as

$$\left.\begin{aligned} \frac{dy}{dx} &= \theta = -\frac{P\lambda^2}{k} e^{-\lambda x} \sin \lambda x, \\ -EI\frac{d^2y}{dx^2} &= M = \frac{P}{4\lambda} e^{-\lambda x}(\cos \lambda x - \sin \lambda x), \\ -EI\frac{d^3y}{dx^3} &= Q = -\frac{P}{2} e^{-\lambda x} \cos \lambda x. \end{aligned}\right\} \tag{e–g}$$

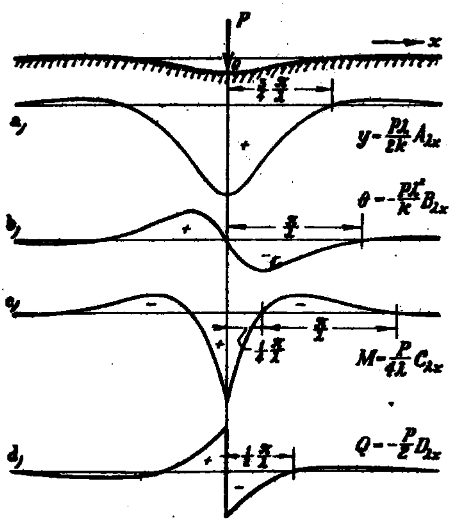

FIG. 5

The curves represented by the equations above are shown in Figure 5. They have all the features of damped waves.* At the point of application of the load ($x = 0$) or, to be precise, infinitely close to the right of it, we have the values $\theta = 0$, $M = P/4\lambda$, and $Q = -P/2$. In the derivation of the general solution for the elastic line (see p. 3) the positive directions were defined for the shearing force Q (positive when acting upward on the left of the elemental section) and for the bending moment M (the moment on the left of the element in the direction of the positive shearing force). As an extension of this convention, we shall regard as positive quantities the downward-acting loading (P), downward deflection (y), and the angular deflection (θ) rotating clockwise. Equations (d–g) give the

* This is the reason why the characteristic λ is sometimes called the *damping factor*.