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SPINORS AND SPACE-TIME

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VOLUME 1
TWO-SPINOR CALCULUS AND RELATIVISTIC FIELDS

R.PENROSE & W.RINDLER

旋量与时空

第1卷

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SPINORS AND SPACE–TIME

Volume 1

Two-spinor calculus and relativistic fields

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Preface

To a very high degree of accuracy, the space–time we inhabit can be taken to be a smooth four-dimensional manifold, endowed with the smooth Lorentzian metric of Einstein's special or general relativity. The formalism most commonly used for the mathematical treatment of manifolds and their metrics is, of course, the tensor calculus (or such essentially equivalent alternatives as Cartan's calculus of moving frames). But in the specific case of four dimensions and Lorentzian metric there happens to exist – by accident or providence – another formalism which is in many ways more appropriate, and that is the formalism of 2-spinors. Yet 2-spinor calculus is still comparatively unfamiliar even now – some seventy years after Cartan first introduced the general spinor concept, and over fifty years since Dirac, in his equation for the electron, revealed a fundamentally important role for spinors in relativistic physics and van der Waerden provided the basic 2-spinor algebra and notation.

The present work was written in the hope of giving greater currency to these ideas. We develop the 2-spinor calculus in considerable detail, assuming no prior knowledge of the subject, and show how it may be viewed either as a useful supplement or as a practical alternative to the more familiar world-tensor calculus. We shall concentrate, here, entirely on 2-spinors, rather than the 4-spinors that have become the more familiar tools of theoretical physicists. The reason for this is that only with 2-spinors does one obtain a practical alternative to the standard vector–tensor calculus, 2-spinors being the more primitive elements out of which 4-spinors (as well as world-tensors) can be readily built.

Spinor calculus may be regarded as applying at a deeper level of structure of space–time than that described by the standard world-tensor calculus. By comparison, world-tensors are less refined, fail to make transparent some of the subtler properties of space–time brought particularly to light by quantum mechanics and, not least, make certain types of mathematical calculations inordinately heavy. (*Their* strength lies in a general applicability to manifolds of arbitrary dimension, rather than in supplying a specific space–time calculus.)

In fact any world-tensor calculation can, by an obvious prescription, be translated entirely into a 2-spinor form. The reverse is also, in a sense, true – and we shall give a comprehensive treatment of such translations later in this book – though the tensor translations of simple spinor manipulations can turn out to be extremely complicated. This effective equivalence may have led some 'sceptics' to believe that spinors are 'unnecessary'. We hope that this book will help to convince the reader that there are many classes of spinorial results about space–time which would have lain undiscovered if only tensor methods had been available, and others whose antecedents and interrelations would be totally obscured by tensor descriptions.

When appropriately viewed, the 2-spinor calculus is also simpler than that of world-tensors. The essential reason is that the basic spin-space is two-complex-dimensional rather than four-real-dimensional. Not only are two dimensions easier to handle than four, but complex algebra and complex geometry have many simple, elegant and uniform properties not possessed by their real counterparts.

Additionally, spinors seem to have profound links with the complex numbers that appear in quantum mechanics.* Though in this work we shall not be concerned with quantum mechanics as such, many of the techniques we describe are in fact extremely valuable in a quantum context. While our discussion will be given entirely classically, the formalism can, without essential difficulty, be adapted to quantum (or quantum-field-theoretic) problems.

As far as we are aware, this book is the first to present a comprehensive development of space–time geometry using the 2-spinor formalism. There are also several other new features in our presentation. One of these is the systematic and consistent use of the *abstract index* approach to tensor and spinor calculus. We hope that the purist differential geometer who casually leafs through the book will not automatically be put off by the appearance of numerous indices. Except for the occasional bold-face upright ones, our indices differ from the more usual ones in being abstract markers without reference to any basis or coordinate system. Our use of abstract indices leads to a number of simplifications over conventional treatments. The use of some sort of index notation seems, indeed, to be virtually essential in order that the necessary detailed manipulations can

* The view that space–time geometry, as well as quantum theory, may be governed by an underlying complex rather than real structure is further developed in the theory of twistors, which is just one of the several topics discussed in the companion volume to the present work: *Spinors and space–time, Vol. 2: Spinor and twistor methods in space–time geometry*, (Cambridge University Press 1985).

be presented in a transparent form. (In an appendix we outline an alternative and equivalent diagrammatic notation which is very valuable for use in private calculations.)

This book appears also to be breaking some new ground in its presentation of several other topics. We provide explicit geometric realizations not only of 2-spinors themselves but also of their various algebraic operations and some of the related topology. We give a host of useful lemmas for both spinor and general tensor algebra. We provide the first comprehensive treatment of (not necessarily normalized) spin-coefficients which includes the compacted spin- and boost-weighted operators ð and þ and their conformally invariant modifications $ð_{\mathscr{C}}$ and $þ_{\mathscr{C}}$. We present a general treatment of conformal invariance; and also an abstract-index-operator approach to the electromagnetic and Yang–Mills fields (in which the somewhat ungainly appearance of the latter is, we hope, compensated by the comprehensiveness of our scheme). Our spinorial treatment of (spin-weighted) spherical harmonics we believe to be new. Our presentation of exact sets of fields as the systems which propagate uniquely away from arbitrarily chosen null-data on a light cone has not previously appeared in book form; nor has the related explicit integral spinor formula (the generalized Kirchhoff–d'Adhémar expression) for representing massless free fields in terms of such data. The development we give for the interacting Maxwell–Dirac theory in terms of sums of integrals described by zig-zag and forked null paths appears here for the first time.

As for the genesis of this work, it goes back to the spring of 1962 when one of us (R.P.) gave a series of seminars on the then-emerging subject of 2-spinors in relativity, and the other (W.R.) took notes and became more and more convinced that these notes might usefully become a book. A duplicated draft of the early chapters was distributed to colleagues that summer. Our efforts on successive drafts have waxed and waned over the succeeding years as the subject grew and grew. Finally during the last three years we made a concerted effort and re-wrote and almost doubled the entire work, and hope to have brought it fully up to date. In its style we have tried to preserve the somewhat informal and unhurried manner of the original seminars, clearly stating our motivations, not shunning heuristic justifications of some of the mathematical results that are needed, and occasionally going off on tangents or indulging in asides. There exist many more rapid and condensed ways of arriving at the required formalisms, but we preferred a more leisurely pace, partly to facilitate the progress of students working on their own, and partly to underline the down-to-earth utility of the subject.

Fortunately our rather lengthy manuscript allowed a natural division into two volumes, which can now be read independently. The essential content of Vol. 1 is summarized in an introductory section to Vol. 2. References in Vol. 1 to Chapters 6–9 refer to Vol. 2.

We owe our thanks to a great many people. Those whom we mention are the ones whose specific contributions have come most readily to mind, and it is inevitable that in the period of over twenty years in which we have been engaged in writing this work, some names will have escaped our memories. For a variety of different kinds of assistance we thank Nikos Batakis, Klaus Bichteler, Raoul Bott, Nick Buchdahl, Subrahmanyan Chandrasekhar, Jürgen Ehlers, Leon Ehrenpreis, Robert Geroch, Stephen Hawking, Alan Held, Nigel Hitchin, Jim Isenberg, Ben Jeffryes, Saunders Mac Lane, Ted Newman, Don Page, Felix Pirani, Ivor Robinson, Ray Sachs, Engelbert Schücking, William Shaw, Takeshi Shirafuji, Peter Szekeres, Paul Tod, Nick Woodhouse, and particularly, Dennis Sciama for his continued and unfailing encouragement. Our thanks go also to Markus Fierz for a remark leading to the footnote on p. 321. Especially warm thanks go to Judith Daniels for her encouragement and detailed criticisms of the manuscript when the writing was going through a difficult period. We are also greatly indebted to Tsou Sheung Tsun for her caring assistance with the references and related matters. Finally, to those people whose contributions we can no longer quite recall we offer both our thanks and our apologies.

Roger Penrose
Wolfgang Rindler

1984

Contents

1
The geometry of world-vectors and spin-vectors

1.1 Minkowski vector space

In this chapter we are concerned with geometry relating to the space of *world-vectors*. This space is called *Minkowski vector space*. It consists of the set of 'position vectors' in the space–time of special relativity, originating from an arbitrarily chosen origin-event. In the curved space–time of general relativity, Minkowski vector spaces occur as the tangent spaces of space–time points (events). Other examples are the space spanned by four-velocities and by four-momenta.

A Minkowski vector space is a four-dimensional vector space $\mathbb{V}$ over the field $\mathbb{R}$ of real numbers, $\mathbb{V}$ being endowed with an orientation, a (bilinear) inner product of signature $(+---)$, and a time-orientation. (The precise meanings of these terms will be given shortly.) Thus, as for any vector space, we have operations of addition, and multiplication by scalars, satisfying

$$\begin{gathered} \boldsymbol{U}+\boldsymbol{V}=\boldsymbol{V}+\boldsymbol{U}, \quad \boldsymbol{U}+(\boldsymbol{V}+\boldsymbol{W})=(\boldsymbol{U}+\boldsymbol{V})+\boldsymbol{W}, \\ a(\boldsymbol{U}+\boldsymbol{V})=a\boldsymbol{U}+a\boldsymbol{V}, \quad (a+b)\boldsymbol{U}=a\boldsymbol{U}+b\boldsymbol{U}, \\ a(b\boldsymbol{U})=(ab)\boldsymbol{U}, \quad 1\boldsymbol{U}=\boldsymbol{U}, \quad 0\boldsymbol{U}=0\boldsymbol{V}=:\boldsymbol{0} \end{gathered} \tag{1.1.1}$$

for all $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}\in\mathbb{V}$, $a, b\in\mathbb{R}$. $\boldsymbol{0}$ is the neutral element of addition. As is usual, we write $-\boldsymbol{U}$ for $(-1)\boldsymbol{U}$, and we adopt the usual conventions about brackets and minus signs, e.g., $\boldsymbol{U}+\boldsymbol{V}-\boldsymbol{W}=(\boldsymbol{U}+\boldsymbol{V})+(-\boldsymbol{W})$, etc.

The four-dimensionality of $\mathbb{V}$ is equivalent to the existence of a *basis* consisting of four linearly independent vectors $\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\in\mathbb{V}$. That is to say, any $\boldsymbol{U}\in\mathbb{V}$ is uniquely expressible in the form

$$\boldsymbol{U}=U^0\boldsymbol{t}+U^1\boldsymbol{x}+U^2\boldsymbol{y}+U^3\boldsymbol{z} \tag{1.1.2}$$

with the *coordinates* $U^0, U^1, U^2, U^3\in\mathbb{R}$; and only $\boldsymbol{0}$ has all coordinates zero. Any other basis for $\mathbb{V}$ must also have four elements, and *any* set of four linearly independent elements of $\mathbb{V}$ constitutes a basis. We often refer to a basis for $\mathbb{V}$ as a *tetrad*, and often denote a tetrad $(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ by $\boldsymbol{g}_i$, where

$$\boldsymbol{t}=\boldsymbol{g}_0, \boldsymbol{x}=\boldsymbol{g}_1, \boldsymbol{y}=\boldsymbol{g}_2, \boldsymbol{z}=\boldsymbol{g}_3. \tag{1.1.3}$$

Then (1.1.2) becomes

$$U = U^0 g_0 + U^1 g_1 + U^2 g_2 + U^3 g_3 = U^{\mathbf{i}} g_{\mathbf{i}}. \tag{1.1.4}$$

Here we are using the Einstein *summation convention*, as we shall henceforth: it implies a summation whenever a *numerical* index occurs twice in a term, once up, once down. Bold-face upright lower-case latin indices $\mathbf{a}, \mathbf{i}, \mathbf{a}_0, \mathbf{a}_1, \hat{\mathbf{a}}$, etc., will always be understood to range over the four values 0, 1, 2, 3. Later we shall also use bold-face upright capital latin letters $\mathbf{A}, \mathbf{I}, \mathbf{A}_0, \mathbf{A}_1, \hat{\mathbf{A}}$, etc., for numerical indices which will range only over the two values 0, 1. Again the summation convention will apply.

Consider two bases for $\mathbb{V}$, say (g_0, g_1, g_2, g_3) and $(g_{\hat{0}}, g_{\hat{1}}, g_{\hat{2}}, g_{\hat{3}})$. Note that we use the 'marked index' notation, in which indices rather than kernel letters of different bases, etc., carry the distinguishing marks (hats, etc.). And indices like $\mathbf{a}, \hat{\mathbf{a}}, \hat{\hat{\mathbf{a}}}$, etc., are as unrelated numerically as $\mathbf{a}, \mathbf{b}, \mathbf{c}$. The reader may feel at first that this notation is unaesthetic but it pays to get used to it; its advantages will becomes apparent later. Now, each vector $g_{\mathbf{i}}$ of the first basis will be a linear combination of the vectors $g_{\hat{\mathbf{i}}}$ of the second:

$$\begin{aligned} g_{\mathbf{i}} &= g_{\mathbf{i}}{}^{\hat{0}} g_{\hat{0}} + g_{\mathbf{i}}{}^{\hat{1}} g_{\hat{1}} + g_{\mathbf{i}}{}^{\hat{2}} g_{\hat{2}} + g_{\mathbf{i}}{}^{\hat{3}} g_{\hat{3}} \\ &= g_{\mathbf{i}}{}^{\hat{\mathbf{j}}} g_{\hat{\mathbf{j}}}. \end{aligned} \tag{1.1.5}$$

The 16 numbers $g_{\mathbf{i}}{}^{\hat{\mathbf{j}}}$ form a (4 × 4) real non-singular matrix. Thus $\det(g_{\mathbf{i}}{}^{\hat{\mathbf{j}}})$ is non-zero. If it is *positive*, we say that the tetrads $g_{\mathbf{i}}$ and $g_{\hat{\mathbf{i}}}$ have the *same orientation*; if *negative*, the tetrads are said to have *opposite orientation*. Note that the relation of 'having the same orientation' is an equivalence relation. For if $g_{\hat{\mathbf{i}}} = g_{\hat{\mathbf{i}}}{}^{\mathbf{j}} g_{\mathbf{j}}$, then $(g_{\hat{\mathbf{i}}}{}^{\mathbf{j}})$ and $(g_{\mathbf{i}}{}^{\hat{\mathbf{j}}})$ are inverse matrices, so their determinants have the same sign; if $g_{\mathbf{i}} = g_{\mathbf{i}}{}^{\hat{\mathbf{j}}} g_{\hat{\mathbf{j}}}$ and $g_{\hat{\mathbf{i}}} = g_{\hat{\mathbf{i}}}{}^{\hat{\hat{\mathbf{j}}}} g_{\hat{\hat{\mathbf{j}}}}$, then the matrix $(g_{\mathbf{i}}{}^{\hat{\hat{\mathbf{j}}}})$ is the product of $(g_{\mathbf{i}}{}^{\hat{\mathbf{j}}})$ with $(g_{\hat{\mathbf{i}}}{}^{\hat{\hat{\mathbf{j}}}})$ and so has positive determinant if both the others have. Thus the tetrads fall into two disjoint equivalence classes. Let us call the tetrads of one class *proper* tetrads and those of the other class *improper* tetrads. It is this selection that gives $\mathbb{V}$ its *orientation.*

The *inner product operation on* $\mathbb{V}$ assigns to any pair U, V of $\mathbb{V}$ a real number, denoted, by $U \cdot V$, such that

$$U \cdot V = V \cdot U, \quad (aU) \cdot V = a(U \cdot V), \quad (U + V) \cdot W = U \cdot W + V \cdot W, \tag{1.1.6}$$

i.e., the operation is symmetric and bilinear. We also require the inner product to have signature $(+ - - -)$. This means that there exists a tetrad (t, x, y, z) such that

$$t \cdot t = 1, \quad x \cdot x = y \cdot y = z \cdot z = -1 \tag{1.1.7}$$

$$t \cdot x = t \cdot y = t \cdot z = x \cdot y = x \cdot z = y \cdot z = 0. \tag{1.1.8}$$

If we denote this tetrad by $g_{\mathbf{i}}$ according to the scheme (1.1.3), then we can rewrite (1.1.7) and (1.1.8) succinctly as

$$g_i \cdot g_j = \eta_{ij}, \tag{1.1.9}$$

where the matrix (η_{ij}) is given by

$$(\eta_{ij}) = (\eta^{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{1.1.10}$$

(The raised-index version η^{ij} will be required later for notational consistency.) We shall call a tetrad satisfying (1.1.9) a ***Minkowski tetrad***. For a given vector space over the real numbers, it is well known (Sylvester's 'inertia of signature' theorem) that for *all* orthogonal tetrads (or 'ennuples' in the *n*-dimensional case), i.e., those satisfying (1.1.8), the number of positive self-products (1.1.7) is invariant.

Given any Minkowski tetrad g_i, we can, in accordance with (1.1.4), represent any vector $U \in \mathbb{V}$ by its corresponding Minkowski coordinates U^i; then the inner product takes the form

$$\begin{aligned} U \cdot V &= (U^i g_i) \cdot (V^j g_j) = U^i V^j (g_i \cdot g_j) \\ &= U^i V^j \eta_{ij} \\ &= U^0 V^0 - U^1 V^1 - U^2 V^2 - U^3 V^3. \end{aligned} \tag{1.1.11}$$

Note that $U \cdot g_j = U^i \eta_{ij}$. Thus,

$$U^0 = U \cdot g_0,\ U^1 = -U \cdot g_1,\ U^2 = -U \cdot g_2,\ U^3 = -U \cdot g_3. \tag{1.1.12}$$

A particular case of inner product is the *Lorentz norm*

$$\|U\| = U \cdot U = U^i U^j \eta_{ij} = (U^0)^2 - (U^1)^2 - (U^2)^2 - (U^3)^2. \tag{1.1.13}$$

We may remark that the inner product can be defined in terms of the Lorentz norm by

$$U \cdot V = \tfrac{1}{2}\{\|U + V\| - \|U\| - \|V\|\}. \tag{1.1.14}$$

The vector $U \in \mathbb{V}$ is called

$$\left.\begin{array}{ll} \textit{timelike} \text{ if} & \|U\| > 0 \\ \textit{spacelike} \text{ if} & \|U\| < 0 \\ \textit{null} \text{ if} & \|U\| = 0. \end{array}\right\} \tag{1.1.15}$$

In terms of its Minkowski coordinates, U is *causal* (i.e., timelike or null) if

$$(U^0)^2 \geqslant (U^1)^2 + (U^2)^2 + (U^3)^2, \tag{1.1.16}$$

with equality holding if U is null. If each of U and V is causal, then applying in succession (1.1.16) and the Schwarz inequality, we obtain

$$|U^0V^0| \geqslant \{(U^1)^2+(U^2)^2+(U^3)^2\}^{\frac{1}{2}}\{(V^1)^2+(V^2)^2+(V^3)^2\}^{\frac{1}{2}}$$
$$\geqslant U^1V^1+U^2V^2+U^3V^3, \tag{1.1.17}$$

Hence unless U and V are both null and proportional to one another, or unless one of them is zero (the only cases in which both inequalities reduce to equalities), then by (1.1.11), the sign of $U \cdot V$ is the same as the sign of U^0V^0. Thus, in particular, no two non-zero causal vectors can be orthogonal unless they are null and proportional.

As a consequence the causal vectors fall into two disjoint classes, such that the inner product of any two non-proportional members of the same classes is *positive* while the inner product of non-proportional members of different classes is *negative*. These two classes are distinguished according to the sign of U^0, the class for which U^0 is positive being the class to which the timelike tetrad vector $t = g_0$ belongs. The *time-orientation* of $\mathbb{V}$ consists in calling *future-pointing* the elements of one of these classes, and *past-pointing* the elements of the other. We often call a future-pointing timelike [null, causal] vector simply a future-timelike [-null, -causal] vector. If t is a future-timelike vector, then the Minkowski tetrad (t, x, y, z) is called *orthochronous*. When referred to an orthochronous Minkowski tetrad, the future-causal vectors are simply those for which $U^0 > 0$. The zero vector, though null, is neither future-null nor past-null. The negative of any future-causal vector is past-causal.

The *space-orientation* of $\mathbb{V}$ consists in assigning 'right-handedness' or 'left-handedness' to the three spacelike vectors of each Minkowski tetrad. This can be done in terms of the orientation and time-orientation of $\mathbb{V}$. Thus the triad (x, y, z) is called *right-handed* if the Minkowski tetrad (t, x, y, z) is both proper and orthochronous, or neither. Otherwise the triad (x, y, z) is *left-handed*. A Minkowski tetrad which is both proper and orthochronous is called *restricted*. Any two of the orientation, time-orientation, and space-orientation of $\mathbb{V}$ determine the third, and if any two are reversed, the third must remain unchanged. When making these choices in the space–time we inhabit, it may be preferable to begin by choosing a triad (x, y, z) and calling it right- or left-handed according to that well-known criterion which physicists use and which is based on the structure of the hand with which most people write.* Similarly statistical physics determines a unique future sense.

* In view of the observed non-invariance of weak interactions under space-reflection (P) and of K^0-decay under combined space-reflection and particle–antiparticle interchange (CP) it is now possible to specify the space-orientation of physical space-time independently of such cultural or physiological considerations: *cf.* Lee and Yang (1956), Wu, Ambler, Hayward Hoppes and Hudson (1957), Lee, Oehme and Yang (1957), Christenson, Cronin, Fitch and Turlay (1964), Wu and Yang (1964); also Gardner (1967) for a popular account.

Minkowski space–time

As we mentioned earlier, Minkowski vector space $\mathbb{V}$ can be regarded as the space of position vectors, relative to an arbitrarily chosen origin, of the points (events) which constitute *Minkowski space–time* $\mathbb{M}$. That space–time is the stage for special relativity theory. None of its points is preferred, and specifically it has no preferred origin: it is invariant under translations, i.e., it is an affine space. The relation between $\mathbb{M}$ and $\mathbb{V}$ can be characterized by the map

$$\text{vec}: \mathbb{M} \times \mathbb{M} \to \mathbb{V} \tag{1.1.18}$$

for which

$$\text{vec}(P, Q) + \text{vec}(Q, R) = \text{vec}(P, R), \tag{1.1.19}$$

whence $\text{vec}(P, P) = 0$ and $\text{vec}(P, Q) = -\text{vec}(Q, P)$. We can regard $\text{vec}(P, Q)$ as the position vector $\overrightarrow{PQ} \in \mathbb{V}$ of Q relative to P, where $P, Q \in \mathbb{M}$. Evidently $\mathbb{V}$ induces by this map a norm, here called the *squared interval* Φ, on any pair of points $P, Q \in \mathbb{M}$:

$$\Phi(P, Q) := \|\text{vec}(P, Q)\| \tag{1.1.20}$$

The standard coordinatization of $\mathbb{M}$, $\mathbb{M} \leftrightarrow \mathbb{R}^4$, where $\mathbb{R}^4$ is the space of quadruples of real numbers, consists of a choice of origin $O \in \mathbb{M}$ and a choice of Minkowski tetrad $g_i = \overrightarrow{OQ_i}$ for $Q_0, Q_1, Q_2, Q_3 \in \mathbb{M}$. Then the coordinates P^0, P^1, P^2, P^3 of any point $P \in \mathbb{M}$ are the coordinates of the vector $\overrightarrow{OP}$ relative to g_i, i.e. $\overrightarrow{OP} = P^i g_i$. From (1.1.19) we find, by putting O for Q, the following coordinates of $\overrightarrow{PR}$ relative to g_i:

$$(\overrightarrow{PR})^i = R^i - P^i, \tag{1.1.21}$$

clearly independently of the choice of origin. Substituting this and (1.1.20) into (1.1.13) yields

$$\Phi(P, Q) = (Q^0 - P^0)^2 - (Q^1 - P^1)^2 - (Q^2 - P^2)^2 - (Q^3 - P^3)^2. \tag{1.1.22}$$

A linear self-transformation of $\mathbb{V}$ which preserves the Lorentz norm – and therefore, by (1.1.14), also the inner product – is called an (*active*) *Lorentz transformation.* If such a transformation preserves both the orientation and time-orientation of $\mathbb{V}$, it is called a *restricted* Lorentz transformation. Clearly the [restricted] Lorentz transformations form a group, and this group is called the [*restricted*] *Lorentz group.* Similarly a self-transformation of $\mathbb{M}$ which preserves the squared interval (no linearity assumption being here needed) is called an (*active*) *Poincaré transformation.* Any such transformation induces a Lorentz transformation on $\mathbb{V}$, and can accordingly also be classified as restricted or not. Again, the restricted Poincaré transformations clearly form a group.*

Any physical experiment going on in the Minkowski space–time of our experience may be subjected to a Poincaré transformation – i.e., rotated in space, translated in space and time, and given a uniform motion – without altering its intrinsic outcome. This is the basis of special relativity theory, and it can be stated without reference to coordinates or to the other laws of physics.

Coordinate change

If not further qualified, Lorentz and Poincaré transformations in this book will be understood to be *active*. But it is sometimes useful to consider 'passive' Lorentz [and Poincaré] transformations. These are transformations of the *coordinate space* $\mathbb{R}^4$, i.e. re-coordinatizations of $\mathbb{V}$ [*or* $\mathbb{M}$]. Any Minkowski tetrad $\boldsymbol{g}_{\mathbf{i}}$ in $\mathbb{V}$ [*or* tetrad $\boldsymbol{g}_{\mathbf{i}}$ and origin O in $\mathbb{M}$] defines a quadruple of coordinates $U^{\mathbf{i}}$ for each U of $\mathbb{V}$ [*or* $U = \overrightarrow{OP}$ of $\mathbb{M}$], with $U = U^{\mathbf{i}}\boldsymbol{g}_{\mathbf{i}}$. A change in this *reference tetrad,* $\boldsymbol{g}_{\mathbf{i}} \mapsto \boldsymbol{g}_{\hat{\mathbf{i}}}$ in $\mathbb{V}$ [*or* of tetrad and origin in $\mathbb{M}$] induces a change in the coordinates for $\mathbb{V}[\mathbb{M}]$. The resulting correspondence

$$G: U^{\mathbf{i}} \mapsto U^{\hat{\mathbf{i}}} \tag{1.1.23}$$

$$[\textit{or } U^{\mathbf{i}} \mapsto U^{\hat{\mathbf{i}}} + K^{\hat{\mathbf{i}}} \text{ with } K^{\hat{\mathbf{i}}} \text{ const.}]$$

is called a *passive Lorentz* [*Poincaré*] *transformation*. It is called *restricted if it can be generated by two restricted Minkowski tetrads* $\boldsymbol{g}_{\mathbf{i}}$ *and* $\boldsymbol{g}_{\hat{\mathbf{i}}}$. For the sake of conciseness, we shall now concentrate on Lorentz transformations, obvious generalizations being applicable to Poincaré transformations.

It the two reference tetrads are related by

$$\boldsymbol{g}_{\mathbf{i}} = g_{\mathbf{i}}{}^{\hat{\mathbf{i}}}\boldsymbol{g}_{\hat{\mathbf{i}}}, \tag{1.1.24}$$

then

$$U = U^{\hat{\mathbf{i}}}\boldsymbol{g}_{\hat{\mathbf{i}}} = U^{\mathbf{i}}\boldsymbol{g}_{\mathbf{i}} = U^{\mathbf{i}}g_{\mathbf{i}}{}^{\hat{\mathbf{i}}}\boldsymbol{g}_{\hat{\mathbf{i}}},$$

and thus the passive transformation (1.1.23) is given explicitly by

$$U^{\hat{\mathbf{i}}} = U^{\mathbf{i}}g_{\mathbf{i}}{}^{\hat{\mathbf{i}}}, \tag{1.1.25}$$

which is evidently linear. It is fully characterized by the matrix $g_{\mathbf{i}}{}^{\hat{\mathbf{i}}}$.

It is often convenient, though slightly misleading, to describe even an

* Note that we use the term 'Lorentz group' here only for the six-parameter *homogeneous* group on Minkowski vector space, while referring to the corresponding ten-parameter inhomogeneous group on Minkowski space–time as the Poincaré group.

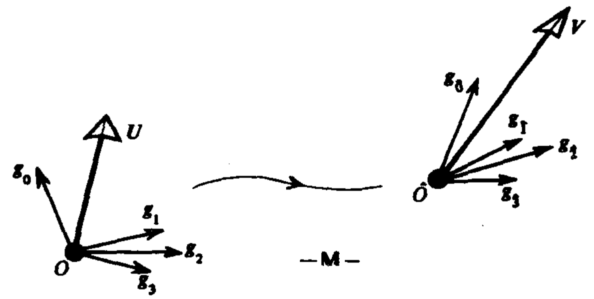

Fig. 1-1. An active Poincaré transformation sends the world vector U at O to a world vector V at $\hat{O}$. If it also sends the tetrad g_i at O to $g_{\hat{\imath}}$ at $\hat{O}$, then the coordinates U^i, of U in g_i, are the same as those, $V^{\hat{\imath}}$, of V in $g_{\hat{\imath}}$, (i.e. $U^i = V^{\hat{\imath}}$). Hence the (reversed) passive transformation induced by $\{g_{\hat{\imath}} \text{ at } \hat{O}\} \mapsto \{g_i \text{ at } O\}$ takes the original coordinates $U^i (= V^{\hat{\imath}})$ of U to the original coordinates V^i of V.

active Lorentz transformation by means of coordinates. (It is slightly misleading because an active Lorentz transformation exists independently of all coordinates, whereas a passive Lorentz transformation does not.) Thus, for a given active Lorentz transformation $L: U \mapsto V$, we can refer both U and its image V to one (arbitrary) Minkowski tetrad $g_{\hat{\imath}}$, whose pre-image under L, let us say, is g_i as in (1.1.24). Since by the assumed linearity of L the expression of V in terms of $g_{\hat{\imath}}$ must be identical with the expression of U in terms of g_i, we then have, from (1.1.25), (see also Fig. 1-1)

$$U^{i} = V^{j} g_{j}^{\hat{\imath}}, \tag{1.1.26}$$

where, in violation of the general rule, we here for once understand summation over the unlike index pair j and $\hat{\jmath}$. We therefore have the following explicit form of the transformation,

$$V^{j} = U^{i} L_{i}^{j}, \tag{1.1.27}$$

where

$$(L_{i}^{j}) = (g_{j}^{\hat{\imath}})^{-1}. \tag{1.1.28}$$

Thus the active Lorentz transformation L that carries g_i into $g_{\hat{\imath}}$ is *formally* equivalent, in its effect on the coordinates of a vector, to the passive Lorentz transformation G^{-1} induced by the passage from $g_{\hat{\imath}}$ to g_i as reference tetrad.

If L is a restricted Lorentz transformation, it clearly carries a restricted Minkowski tetrad into a restricted Minkowski tetrad, and thus the corresponding passive transformation G is restricted also. If, conversely, G is restricted, suppose it is generated by the restricted tetrads g_i and $g_{\hat{\imath}}$; then the corresponding L preserves norms, products, and orientation since, in fact, it preserves coordinates, and thus L is restricted. Now in order for L to preserve inner products we require – from (1.1.11) and (1.1.27), dropping hats –

$$\eta_{ij}L_k{}^iL_l{}^j = \eta_{kl}. \tag{1.1.29}$$

Regarding this as a matrix equation, we see that $\det(L_i{}^j) = \pm 1$. The condition for L to be restricted is then seen to be

$$\det(L_i{}^j) = 1, \qquad L_0{}^0 > 0. \tag{1.1.30}$$

Because of (1.1.28), the same conditions apply to the matrix of a passive restricted Lorentz transformation. They can, of course, also be derived directly from the definitions:

$$\eta_{ij}g_{\hat{i}}{}^i g_{\hat{j}}{}^j = \eta_{\hat{i}\hat{j}}, \quad \det(g_{\hat{i}}{}^i) = 1, \quad g_{\hat{0}}{}^0 > 0. \tag{1.1.31}$$

1.2 Null directions and spin transformations

In §1.1 the conventional representation of a world-vector $\boldsymbol{U}$ in terms of *Minkowski* coordinates was considered. Now we examine another way of representing world-vectors by coordinates. In particular, we shall obtain a coordinatization of the null cone (i.e., the set of null vectors) in terms of complex numbers. This will lead us to the concept of a spin-vector.

To avoid unnecessary indices, we write T, X, Y, Z for the coordinates U^0, U^1, U^2, U^3 of $\boldsymbol{U}$ with respect to a restricted Minkowski tetrad $(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$:

$$\boldsymbol{U} = T\boldsymbol{t} + X\boldsymbol{x} + Y\boldsymbol{y} + Z\boldsymbol{z}. \tag{1.2.1}$$

For *null* vectors the coordinates satisfy

$$T^2 - X^2 - Y^2 - Z^2 = 0. \tag{1.2.2}$$

Often we wish to consider just the null *directions*, say at the origin O of (Minkowski) space-time. Note that $\pm\boldsymbol{U}$ will be considered to have unequal (namely, opposite) directions. The abstract space whose elements are the future [past] null directions we call $\mathscr{S}^+$ [$\mathscr{S}^-$]. These two spaces can be *represented* in any given coordinate system (T, X, Y, Z) by the intersections S^+ [S^-] of the future [past] null cone (1.2.2) with the hyperplanes $T = 1$ [$T = -1$]. In the Euclidean (X, Y, Z)-space $T = 1$ [$T = -1$], S^+ [S^-] is a sphere with equation*

$$x^2 + y^2 + z^2 = 1. \tag{1.2.3}$$

(See Fig. 1-2) Of course, the direction of *any* vector (1.2.1) through O

* We here reserve lower case letters x, y, z for coordinates on S^+ and S^-