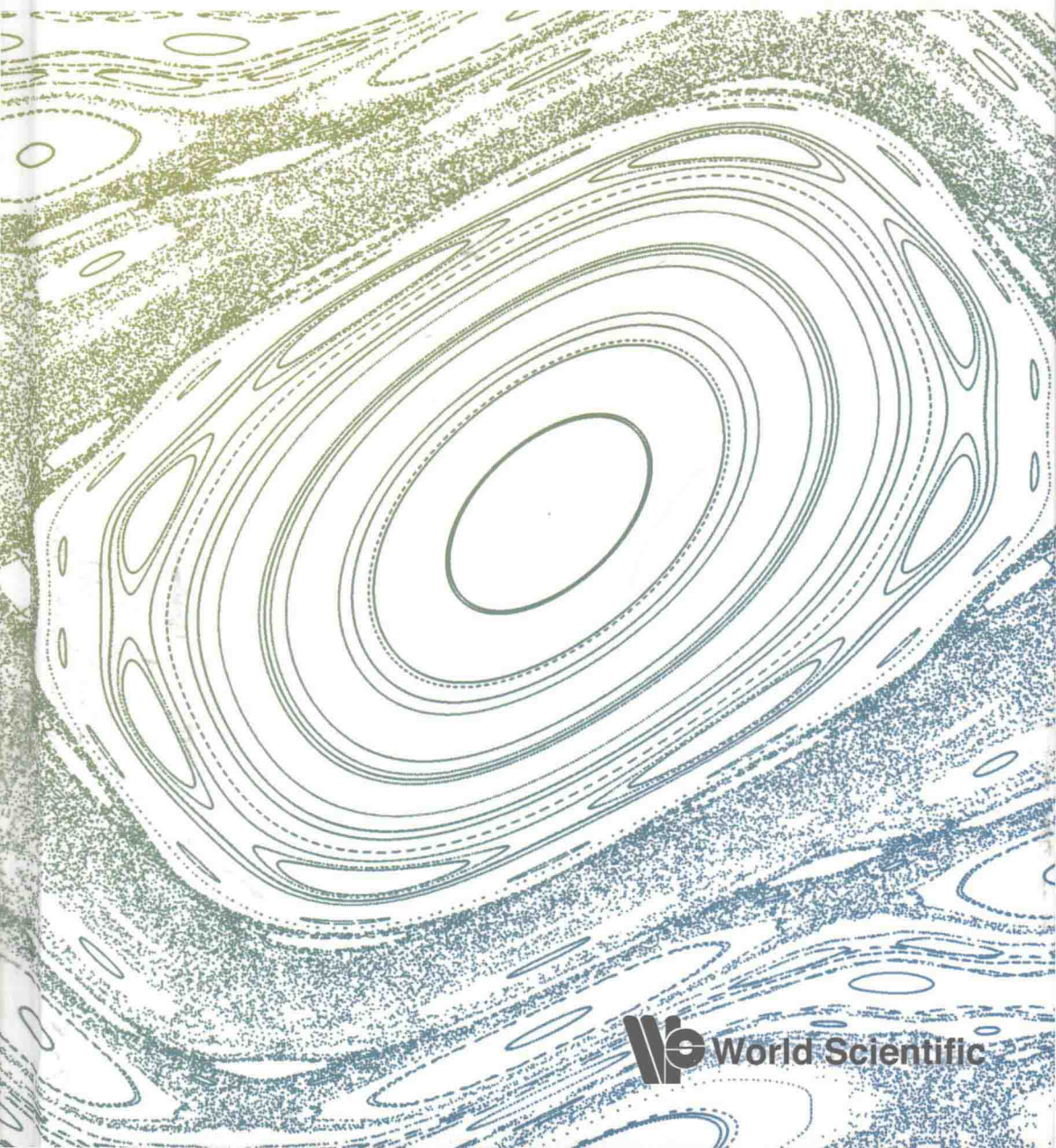


# THE KAM STORY

A Friendly Introduction to the Content, History, and  
Significance of Classical Kolmogorov–Arnold–Moser Theory

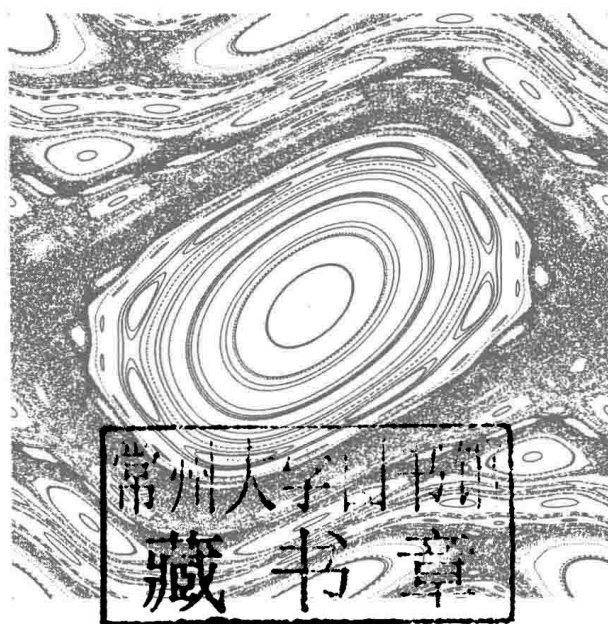
H Scott Dumas



 World Scientific

# THE KAM STORY

A Friendly Introduction to the Content, History, and  
Significance of Classical Kolmogorov–Arnold–Moser Theory



H Scott Dumas

University of Cincinnati, USA

 **World Scientific**

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

*Published by*

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

### **Library of Congress Cataloging-in-Publication Data**

Dumas, H. Scott, author.

The KAM story : a friendly introduction to the content, history, and significance of classical Kolmogorov-Arnold-Moser theory / by H. Scott Dumas (University of Cincinnati, USA).

pages cm

Includes bibliographical references and index.

ISBN 978-9814556583 (alk. paper)

1. Kolmogorov-Arnold-Moser Theorem. 2. Science--History. I. Title.

Q172.5.C45D86 2014

003'.85--dc23

2014000204

### **British Library Cataloguing-in-Publication Data**

A catalogue record for this book is available from the British Library.

Cover illustration courtesy of Ben Vaughan, showing KAM circles and chaotic orbits in the Chirikov standard map.

Figures by the author, unless otherwise indicated.

Copyright © 2014 by World Scientific Publishing Co. Pte. Ltd.

*All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the publisher.*

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

Printed in Singapore by World Scientific Printers.

# THE KAM STORY

A Friendly Introduction to the Content, History, and  
Significance of Classical Kolmogorov–Arnold–Moser Theory



To Warren Farnholtz



# Preface

In many ways classical mechanics serves as the bedrock of physical science, yet surprisingly, it has crucial features that are not widely known. Many people know something about ‘chaos theory’—how mathematical models of certain deterministic classical systems fail to predict the evolution of those systems in a practical sense. If they’re interested in history, they also know that much of chaos theory was understood by mathematicians almost a century before it was popularized by way of computer models in the last third of the 20th century. But there is a deeper, more interesting story that is not well known outside a circle of experts, and the aim of this book is to tell this story to a wider audience.

The story in a nutshell is this: Right from the start, after enunciating his laws of mechanics and gravitation, Isaac Newton ran into difficulties using those laws to describe the motion of three bodies moving under mutual gravitational attraction (the so-called ‘three body problem’). For the next two centuries, these difficulties resisted solution, as the best minds in mathematics and physics concentrated on solving other, increasingly complex model systems in classical mechanics (in the abstract mathematical setting, to ‘solve’ a system means showing that its trajectories move linearly on so-called ‘invariant tori’). But toward the end of the 19th century, using his own new methods, Henri Poincaré confronted Newton’s difficulties head-on and discovered an astonishing form of ‘unsolvability,’ or chaos, at the heart of the three body problem. This in turn led to a paradox. According to Poincaré and his followers, most classical systems should be chaotic; yet observers and experimentalists did not see this in nature, and mathematicians working with model systems could not (quite) prove it to be true either. The paradox persisted for more than a half-century, until Andrey Kolmogorov unraveled it by announcing that, against all expect-



tation, many of the invariant tori from solvable systems remain intact in chaotic systems. These tori make most systems into hybrids—they are a strange, fractal mixture of regularity and chaos. This stunning announcement was later affirmed with rigorous mathematical proofs by Vladimir Arnold and Jürgen Moser, and the names Kolmogorov, Arnold, and Moser were combined in the acronym KAM, by which the theory has since been known. Thus the true picture of classical mechanics—often thought to have been essentially sketched in the 17th century—was not complete until the latter part of the 20th century. And although the mathematical theory is indeed mostly complete, certain applications to problems in physics (especially in celestial and statistical mechanics) have been developed only with great difficulty, and some remain controversial and uncertain even today.

To compare the practical impact of KAM theory to that of relativity or quantum theory is not realistic (to be frank, the practical impact of KAM theory has been limited). Yet in the history of ideas and the philosophy of science, it is not a stretch to rank KAM theory alongside the revolutions in modern physics. But KAM theory—and the paradox that precipitated it—also had the misfortune of playing out over roughly the same interval during which the revolutions of modern physics took place. Not surprisingly, in that period, physicists abandoned classical mechanics to the few hardy mathematicians who remained interested in it. The physicists returned with wondrous stories of their exploits in quantum mechanics, relativity, and nuclear physics. The time has come for mathematicians to tell their tales from this period in a broad setting, too.

When I asked specialists why none of them had yet written a broad overview of KAM theory, they invariably answered that, with several different ‘schools’ having descended from the original founders of the theory, it would be awkward for any one individual to take up that task. In other words, KAM theory is still slightly controversial, and the experts are understandably touchy about each other’s contributions. Since I don’t belong to any particular school, I am prepared to step into the breach, or break the ice. I hope the experts will follow me, not with pitchforks, but with first-hand accounts, corrections, and further detail.

*H.S. Dumas, December 2013*

## Acknowledgments

Like its subject, this book has long roots, and there are many people to thank for its development. My interest in history of science began early and was fueled in my undergraduate days by contact with Albert Van Helden and Salomon Bochner. I heard of KAM theory from Asim Barut at the beginning of graduate school, and was inspired to learn more through discussions with my mentors Jim Ellison, Bill Sáenz, Tom Kyner, Vageli Coutsiás, François Golse, and especially Pierre Lochak, who taught me much of what I know about the subject and pointed me toward the Russian and Italian literature. Later, I had the good fortune to meet and talk with Jürgen Moser and Vladimir Arnold, along with some of their students, which only magnified my interest.

Once I formed the idea of this book, I was strongly encouraged by Hildeberto Cabral, Jie Chen, Chuck Wells, and especially Teresa Stuchi. When I needed to find more detail about Weierstrass and Poincaré, Alain Albouy directed me to original sources; he and Alain Chenciner later helped me understand those sources in context. Many people looked at early versions of the draft and encouraged me, including Mathias Vogt, Ken Meyer, Qiu Dong Wang, Patricia Yanguas, Jesús Palacián, Klaus Heinemann, Ning Zhong, and Bing Yu Zhang. Once written, the manuscript would not have found a good publisher without the key endorsements of Reuben Hersh, Jacques Féjoz, and Steve Wiggins.

I thank the following people for looking over parts of the manuscript and suggesting references or changes, or correcting errors of fact: Dave Levermore, Pierre Lochak, Jacques Laskar, Jacques Féjoz, Jürgen Pöschel, Jean-Pierre Marco, Steve Wiggins, Bruce Pourciau, Rafael de la Llave, Florin Diacu, Jim Murdock, Lawrence C. Evans, and Luigi Chierchia, whose e-mail tutorials were particularly enlightening and helpful. I'm especially

grateful to those who read through and helped edit or correct the entire manuscript, including Heinz Hanßmann, Jim Ellison, Alain Albouy, and Armelle J.F. Clark. Much thanks also to Carles Simó and Ben Vaughan for providing numerically plotted illustrations. Perhaps more than anyone else, I'm indebted to Mikhail Sevryuk for his critical and constructive reading of the book. I also thank him—along with Phil Korman and Leonid Slavin—for help with transliterations and other things Russian that were unfamiliar to me.

Finally, I thank the Charles Phelps Taft Research Center for its support, Nancy Diemler for assistance with  $\text{\LaTeX}$  and other practical matters, and my editors Lai Fun Kwong and Rok Ting Tan at World Scientific for their invaluable assistance.

# Contents

<i>Preface</i>	vii
<i>Acknowledgments</i>	ix
1. Introduction	1
1.1 What this book is, and how it came about . . . . .	2
1.2 Representative quotations and commentary . . . . .	4
1.3 Remarks on the style and organization of this book . . . .	7
2. Minimum Mathematical Background	9
2.1 Dynamical systems . . . . .	9
2.2 Hamiltonian systems . . . . .	13
2.2.1 Two pictures for Hamiltonian systems . . . . .	15
2.2.2 What does it mean to ‘solve’ a Hamiltonian system?	19
2.2.3 Completely integrable Hamiltonian systems . . . .	20
2.2.4 Resonant and nonresonant tori . . . . .	25
2.2.5 A first introduction to the idea of nondegeneracy	26
3. Leading Up to KAM: A Sketch of the History	29
3.1 The planets lead the way . . . . .	29
3.2 Newton, Poincaré, and the most romantic view of KAM .	30
3.3 A more sober view . . . . .	31
3.4 The $n$ body problem . . . . .	32
3.5 The stability problem . . . . .	33
3.6 Toward the modern era—integrability and its vulnerabilities	35
3.7 Weierstrass, Poincaré, and the King Oscar prize . . . . .	38
3.8 Aftermath of the prize: the seeds of change are sown . . .	40

3.9	A quick sketch of Poincaré and his work . . . . .	40
3.10	HPT: ‘The fundamental problem of dynamics’ . . . . .	45
3.11	From small divisors to nonintegrability and chaos . . . . .	46
3.12	The post-Poincaré era . . . . .	52
3.12.1	Poincaré’s legacy in dynamics . . . . .	52
3.12.2	The chaos debate . . . . .	56
3.12.3	Ergodic theory . . . . .	57
3.12.4	Over-indulgence in chaos . . . . .	58
3.12.5	Paradox and a long crisis in mechanics . . . . .	60
4.	KAM Theory . . . . .	63
4.1	C.L. Siegel and A.N. Kolmogorov: Small divisors overcome . . . . .	63
4.2	Kolmogorov’s discovery of persistent invariant tori . . . . .	67
4.3	A closer look at the convergence scheme . . . . .	69
4.3.1	An overview of the scheme . . . . .	70
4.3.2	Technical issues . . . . .	72
4.4	Chronology of Arnold’s and Moser’s work . . . . .	76
4.4.1	Arnold’s chronology . . . . .	76
4.4.2	Moser’s chronology . . . . .	78
4.5	A prototype KAM theorem . . . . .	79
4.6	Early versions of the KAM theorem . . . . .	80
4.7	More recent results that are optimal (or nearly so) . . . . .	84
4.8	Further approaches and results . . . . .	87
5.	KAM in Context: Questions, Consequences, Significance . . . . .	93
5.1	A quick overview of KAM theory in prose and pictures . . . . .	93
5.1.1	A cartoon summary of KAM theory . . . . .	93
5.1.2	Hegel’s last laugh . . . . .	95
5.1.3	The big historical picture . . . . .	96
5.2	Pros and cons, the myths of detractors and enthusiasts . . . . .	99
5.2.1	A list of pros and cons . . . . .	99
5.2.2	Discussion . . . . .	100
5.3	‘Sociological’ issues . . . . .	105
5.3.1	Why did it take so long? . . . . .	105
5.3.2	Why so few Americans? . . . . .	106
5.3.3	Did Kolmogorov prove KAM? . . . . .	107
5.3.4	How hard is the proof? . . . . .	108
5.4	How much celebration is called for? . . . . .	110

5.4.1	Quick summary of the usual arguments . . . . .	110
5.4.2	A plea for KAM theory in classical mechanics . .	111
6.	Other Results in Hamiltonian Perturbation Theory (HPT)	115
6.1	Geometric HPT: KAM, cantori, & Aubry-Mather theory .	116
6.2	Classical HPT: Nekhoroshev theory . . . . .	119
6.2.1	Nekhoroshev's theorem . . . . .	119
6.2.2	Brief history of Nekhoroshev theory & applications	121
6.2.3	Remarks on the proofs in Nekhoroshev theory . .	125
6.3	Instability in HPT . . . . .	126
6.3.1	The Chirikov regime and standard map . . . . .	127
6.3.2	The Nekhoroshev regime and Arnold diffusion . .	131
7.	Physical Applications	147
7.1	Stability of the solar system (or not?) . . . . .	147
7.1.1	KAM theory applied to the $n$ body problem . . .	148
7.1.2	Specialized results for subsystems . . . . .	155
7.1.3	The physical solar system . . . . .	156
7.2	Ramifications in statistical mechanics . . . . .	159
7.2.1	About Boltzmann . . . . .	160
7.2.2	The ergodic hypothesis . . . . .	162
7.2.3	Equipartition, FPU, & the ultraviolet catastrophe	172
7.3	Other applications of KAM in physics . . . . .	177
7.3.1	The generic application: elliptic equilibria . . . .	177
7.3.2	Stability of charged particle motions . . . . .	178
7.3.3	More exotic applications . . . . .	179
Appendix A	Kolmogorov's 1954 paper	181
Appendix B	Overview of Low-dimensional Small Divisor Problems	187
B.1	The linearization problem . . . . .	187
B.1.1	From Schröder's equation to Siegel's problem . . .	187
B.1.2	Refinements and optimality for Siegel's problem .	189
B.2	Mappings of the circle . . . . .	190
Appendix C	East Meets West — Russians, Europeans, Americans	193
C.1	Cultural stereotypes in mathematics . . . . .	194
C.2	Cultural and stylistic tensions . . . . .	195

C.3	Cultural cross-currents in KAM theory . . . . .	196
Appendix D	Guide to Further Reading . . . . .	199
D.1	General references on KAM . . . . .	199
D.1.1	Original KAM articles, and priority . . . . .	199
D.1.2	Accessible proofs of KAM theorems . . . . .	200
D.1.3	Books on KAM theory (what books?) . . . . .	201
D.1.4	Reviews, monographs, & book chapters on KAM . . . . .	202
D.1.5	Expository, historical, & other sources on KAM . . . . .	204
D.2	Mathematical background . . . . .	205
D.2.1	Dynamical systems and ODEs . . . . .	205
D.2.2	Classical mechanics and Hamiltonian dynamics . . . . .	206
D.2.3	Ergodic theory . . . . .	208
D.3	Chaos theory . . . . .	208
D.3.1	The popular side of chaos . . . . .	208
D.3.2	A chaos debate . . . . .	209
D.3.3	The aftermath of popular chaos theory . . . . .	212
D.4	History . . . . .	213
D.4.1	The special nature of history of math & physics . . . . .	213
D.4.2	Early history of mathematics and astronomy . . . . .	213
D.4.3	Between Newton and Poincaré . . . . .	214
D.4.4	Weierstrass and Poincaré's time . . . . .	214
D.4.5	The Painlevé conjecture & the $n$ body problem . . . . .	215
D.4.6	The Soviet & Russian schools of dynamical systems . . . . .	216
D.4.7	History of dynamical systems in general . . . . .	216
D.5	Biography . . . . .	216
D.5.1	General biographical sources . . . . .	217
D.5.2	The principals . . . . .	217
D.6	Applications of KAM (and Nekhoroshev) theory . . . . .	219
D.6.1	Applications to celestial mechanics; stability . . . . .	219
D.6.2	Applications to statistical mechanics, ergodic theory . . . . .	220
D.6.3	Other applications . . . . .	222
D.7	Mathematical topics related to classical KAM theory . . . . .	222
D.7.1	Low-dimensional small divisor problems . . . . .	223
D.7.2	Aubry-Mather & weak KAM theory, KAM for PDE . . . . .	223
D.7.3	Nekhoroshev theory . . . . .	224
D.7.4	Arnold diffusion . . . . .	225

D.8 Culture, philosophy, Bourbaki, etc. . . . .	225
Appendix E Selected Quotations	229
Appendix F Glossary	235
<i>Bibliography</i>	315
<i>Index</i>	351



