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R. E. Edwards

Fourier Series
A Modern Introduction
Volume 2

Second Edition

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PREFACE TO THE SECOND (REVISED) EDITION OF VOLUME 2

Apart from a number of minor corrections and changes, a substantial reformulation and up-dating of Chapters 14 and 15 has taken place. This reformulation and up-dating is a major and very welcome contribution from my friend and colleague, Dr J.W. Sanders, to whom I express my sincere thanks. His efforts have produced a much better result than I could have achieved on my own. Warm thanks are also due to Dr Jo Ward, who checked some of the revised material.

New Sections 16.9 and 16.10 have also been added.

The bibliography has been expanded and brought up to date, though it is still not exhaustive.

In spite of these changes, the third paragraph in the Preface to the revised edition of Volume 1 is applicable here. What has been accomplished here is not a complete account of developments over the past 15 years; such an account would require many volumes. Even so, it may assist some readers who wish to appraise some of these developments. More ambitious readers should consult *Mathematical Reviews* from around Volume 50 onwards.

R. E. E.

CANBERRA, September 1981

PREFACE TO VOLUME 2

The substance of the first three paragraphs of the preface to Volume 1 of *Fourier Series: A Modern Introduction* applies equally well to this second volume. To what is said there, the following remarks should be added.

Volume 2 deals on the whole with the more modern aspects of Fourier theory, and with those facets of the classical theory that fit most naturally into a function-analytic garb. With their introduction to distributional concepts and techniques and to interpolation theorems, respectively, Chapters 12 and 13 are perhaps the most significant portions of Volume 2. From a pedagogical viewpoint, the carefully detailed discussion of Marcinkiewicz's interpolation theorem will, it is hoped, go some way toward making this topic more accessible to a beginner.

A major portion of Chapter 11 is devoted to the elements of Banach algebra theory and its applications in harmonic analysis. In Chapter 16 there appears what is believed to be the first reasonably connected introductory account of multiplier problems and related matters.

For the purposes of a short course, one might be content to cover Section 11.1, the beginning of Section 11.2, Section 11.4, Chapter 12 up to and including Section 12.10, Chapter 13 up to and including Section 13.6, Chapter 14, and Sections 15.1 to 15.3. Much of Chapters 13 to 15 is independent of Chapters 11 and 12, or is easily made so. While severe pruning might lead to a tolerable excision of Section 11.4, which is required but rarely in subsequent chapters, it would be a pity thus to omit all reference to Banach algebras.

I at one time cherished the hope of including in this volume a list of current research problems, but the available space will not accommodate such a list together with the necessary explanatory notes. The interested reader may go a long way toward repairing this defect by studying some of the articles appearing in [Bi] (see, most especially, pp. 351–354 thereof).

The cross-referencing system is as follows. With the exception of references to the appendixes, the numerical component of every reference to either volume appears in the form $a \cdot b \cdot c$, where a, b, and c are positive integers; the material referred to appears in Volume 1 if and only if $1 \leqslant a \leqslant 10$. In the case of references to the appendixes, all of which appear in Volume 1, a Roman numeral "I" has been prefixed as a reminder to the reader; thus, for example, "I,B.2.1" refers to Appendix B.2.1 in Volume 1.

An understanding of the main topics discussed in this book does not, I hope, hinge upon repeated consultation of the items listed in the bibliography. Readers with a limited aim should find strictly necessary only an occasional reference to a few of the book listed. The remaining items, and especially the numerous research papers mentioned, are listed as an aid to those readers who wish to pursue the subject beyond the limits reached in this book; such readers must be prepared to make the very considerable effort called for in making an acquaintance with current research literature. A few of the research papers listed cover developments that came to my notice too late for mention in the main text. For this reason, any attempted summary in the main text of the current standing of a research problem should be supplemented by an examination of the bibliography and by scrutiny of the usual review literature.

Finally, I take this opportunity to renew all the thanks expressed in the preface to Volume 1, placing special reemphasis on those due to Professor Edwin Hewitt for his sustained interest and help, to Dr. Garth Gaudry for his contributions to Chapter 13, and to my wife for her encouragement and help with the proofreading. My thanks for help in the latter connection are extended also to my son Christopher.

CANBERRA, 1967 *R. E. E.*

CONTENTS

CHAPTER 11

Spans of Translates. Closed Ideals. Closed Subalgebras. Banach Algebras

The first three sections of this chapter are devoted to some topics mentioned earlier, namely, the study of closed invariant subspaces and closed ideals [mentioned in 2.2.1 and 3.1.1(g)], and that of closed subalgebras [mentioned in 3.1.1(e) and (f)]. Throughout the discussion **E** will denote any one of the convolution algebras $\mathbf{L}^p$ $(1 \leqslant p < \infty)$ or **C** (see 3.1.1, 3.1.5, and 3.1.6) and we shall consider closed invariant subspaces, closed ideals, and closed subalgebras in **E**. The cases $\mathbf{E} = \mathbf{C}^k$ and $\mathbf{E} = \mathbf{L}^\infty$ could also be treated similarly, provided that in the last case one considered $\mathbf{L}^\infty$ with its so-called weak topology, in which a sequence or net (f_i) converges to f if and only if

$$\lim_i \frac{1}{2\pi} \int f_i g \, dx = \frac{1}{2\pi} \int fg \, dx$$

is true for each $g \in \mathbf{L}^1$. Compare I, B.1.7 and I, C.1.

For any compact group, Abelian or not, the structure theory for closed invariant subspaces and closed ideals is simple. For the group T the details are fully elucidated in 11.2.1. By contrast, except for the case $\mathbf{E} = \mathbf{L}^2$, the structure of closed subalgebras is not yet fully describable, even for the group T.

Subsections 11.2.3 and 11.2.4 are included on "cultural" grounds and are intended to show how the relatively simple problems treated in 11.2.1 and 11.2.2 lead to ones of considerable complexity and interest when the compact group T is replaced by a noncompact group such as R. (These subsections are not essential to an understanding of the rest of the book.) The relevant problems for the dual group Z are mentioned briefly in 11.2.5.

Section 11.3 is devoted to the problem of closed subalgebras in **E**.

The final section of this chapter (11.4) is devoted to a few of the fundamentals of commutative Banach algebra theory and some of its applications to harmonic analysis. When applications are made to the algebras **E** mentioned above, we find that the topics mentioned in Section 4.1 undergo natural development. Applications to other algebras will also be made and will provide proofs of results stated in Section 10.6.

Section 11.4 is in no sense a balanced introduction to the study of Banach algebras. References for further reading will be given in due course.

11.1 Closed Invariant Subspaces and Closed Ideals

By a *closed invariant subspace* of **E** is meant a linear subspace **V** of **E** which is (1) closed for the normal topology of **E** (see 2.2.4), and (2) invariant under translation, in the sense that $f \in \mathbf{V}$ entails $T_a f \in \mathbf{V}$ for all $a \in T$. (Compare the definition of invariant subspaces given in 2.2.1.)

Each $f \in \mathbf{E}$ is contained in a smallest closed invariant subspace $\bar{\mathbf{V}}_f$, which is none other than the closure in **E** of the invariant subspace $\mathbf{V}_f$ generated by f (as defined in 2.2.1). The reader will note that $\bar{\mathbf{V}}_f$ depends in general on the ambient space **E**: for example, if f is continuous, the closure of $\mathbf{V}_f$ in $\mathbf{L}^p$ will in general be strictly larger than the closure of $\mathbf{V}_f$ in **C**. Despite this, we do not think it necessary to complicate the notation accordingly.

In view of the fact that **E** is an algebra under convolution, we follow the usual algebraic terminology by describing as an *ideal* in **E**, a linear subspace **I** of **E** with the property that $f * g \in \mathbf{I}$ whenever $f \in \mathbf{I}$ and $g \in \mathbf{E}$. A *closed ideal* in **E** is an ideal in **E** which is also a closed subset of **E**.

As will be seen in 11.1.2, the closed invariant subspaces of **E** and the closed ideals in **E** are exactly the same things (although the invariant subspaces and the ideals are *not* the same things).

11.1.1. If $f \in \mathbf{E}$, then $\hat{f}(n)e_n \in \bar{\mathbf{V}}_f$ for all $n \in Z$.

Proof. Direct computation shows that

$$\hat{f}(n)e_n = e_n * f.$$

Since $e_n \in \mathbf{L}^1$, the assertion follows from 3.1.9. For an alternative proof, see Exercise 11.5. Yet another type of proof is described in 11.2.2.

11.1.2. A subset of **E** is a closed invariant subspace of **E** if and only if it is a closed ideal in **E**. (Compare with 3.2.3.)

Proof. (1) Let **I** be a closed ideal in **E**. We wish to show that **I** is translation-invariant. For this purpose, we utilize an argument appearing in 3.2.3. Choose an approximate identity $(k_n)_{n=1}^{\infty}$ comprised of elements of **E** (for example, the Fejér kernels introduced in Section 5.1). Since **I** is an ideal, $(T_a k_n) * f \in \mathbf{I}$ for all $n \geq 1$ and all $f \in \mathbf{I}$. But $(T_a k_n) * f = T_a(k_n * f)$ by 3.1.2, and $\lim_{n\to\infty} k_n * f = f$ in **E** by 3.2.2. Therefore $\lim_{n\to\infty} T_a(k_n * f) = T_a f$ in **E**. **I** being closed, it follows that $T_a f \in \mathbf{I}$. This shows that **I** is translation-invariant and is therefore a closed invariant subspace of **E**.

(2) Let **V** be a closed invariant subspace of **E**. In order to prove that **V** is a closed ideal in **E**, it suffices to show that $f * g \in \mathbf{V}$ whenever $f \in \mathbf{V}$ and $g \in \mathbf{E}$. In doing this we may, since **V** is closed in **E** and since the trigonometric

polynomials are everywhere dense in **E** (see 2.4.4), assume that g is a trigonometric polynomial; see 3.1.6. In that case, however, $f * g$ is a finite linear combination of terms $\hat{f}(n)e_n$, and 11.1.1 shows at once that $f * g \in \mathbf{\bar{V}}_f$. Finally, since $f \in \mathbf{V}$, $\mathbf{\bar{V}}_f \subset \mathbf{V}$, and therefore $f * g \in \mathbf{V}$. The proof is complete.

11.1.3. **Remarks.** (1) It has been noted in 3.2.3 that **E** is a module over $\mathbf{L}^1$; and in Section 12.7 it will appear that **E** is even a module over the superspace **M** of $\mathbf{L}^1$ composed of all Radon measures. It is quite simple to verify that the closed submodules of **E** (qua module over $\mathbf{L}^1$ or over **M**) are exactly the closed ideals in **E**.

(2) The reader will take care to remember that 11.1.2 is established only for the choices of **E** mentioned at the outset of this chapter; it is not true in all cases of interest. For example, if $\mathbf{L}^\infty$ is taken with its normed topology, there are closed ideals in the convolution algebra $\mathbf{L}^\infty$ that are not translation-invariant; see Exercises 11.22 and 11.23. Theorem 11.1.2 is also false for the measure algebra **M** introduced in Section 12.7; see Exercise 12.45.

11.2 The Structure of Closed Ideals and Related Topics

It can now be shown that a closed ideal **I** in **E** is characterized completely in terms of the common zeros of the Fourier transforms of elements of **I**.

For any $f \in \mathbf{E}$, we denote by Z_f the set of $n \in Z$ for which $\hat{f}(n) = 0$; and for any subset **S** of **E** we write

$$Z_{\mathbf{S}} = \bigcap \{Z_f : f \in \mathbf{S}\}.$$

11.2.1. Let **I** be any closed ideal in **E**, and let $f \in \mathbf{E}$. Then $f \in \mathbf{I}$ if and only if $Z_f \supset Z_{\mathbf{I}}$.

Proof. Obviously, $Z_f \supset Z_{\mathbf{I}}$ whenever $f \in \mathbf{I}$. Suppose conversely that $f \in \mathbf{E}$ and $Z_f \supset Z_{\mathbf{I}}$; we have to show that $f \in \mathbf{I}$. Let $n \notin Z_{\mathbf{I}}$ and choose $g \in \mathbf{I}$ such that $\hat{g}(n) \neq 0$. By 11.1.1, $e_n \in \mathbf{\bar{V}}_g$; and by 11.1.2, $\mathbf{\bar{V}}_g \subset \mathbf{I}$. Thus $e_n \in \mathbf{I}$, and this for any $n \notin Z_{\mathbf{I}}$. A fortiori, $e_n \in \mathbf{I}$ for any n for which $\hat{f}(n) \neq 0$. Now 6.1.1 shows that f is the limit in **E** of finite linear combinations of exponentials e_n with n restricted by the condition $\hat{f}(n) \neq 0$. Since **I** is a closed linear subspace of **E**, it appears that $f \in \mathbf{I}$, as was to be proved.

Remarks. (1) In view of 11.1.2, 11.2.1 may be reformulated in the following way. Let **V** be a closed invariant subspace of **E** and put $S = Z \backslash Z_{\mathbf{V}}$; then **V** is identical with the closed linear subspace of **E** generated by $\{e_n : n \in S\}$. In brief, **V** is generated (as a closed linear subspace, a closed invariant subspace, or a closed ideal) by the continuous characters it contains.

The equivalence of the two versions depends upon 6.1.1. As usual, the result remains true for $\mathbf{E} = \mathbf{L}^\infty$, provided the weak topology is used throughout; in this connection it is useful (although not essential) to note that $\lim_{N \to \infty} \sigma_N f = f$ weakly in $\mathbf{L}^\infty$ whenever $f \in \mathbf{L}^\infty$.

(2) In 11.2.1 it is essential that the ideal **I** be assumed to be closed. For example, if **I** is any everywhere dense and nonclosed ideal in **E**, then $Z_{\mathbf{I}} = \varnothing = Z_{\mathbf{E}}$ but $\mathbf{I} \neq \mathbf{E}$. In such cases there is no known simple structure theorem.

(3) For a study of projections onto closed invariant subspaces of $\mathbf{L}^p(G)$, where G is a noncompact group, see Rosenthal [1].

11.2.2. **The Hahn-Banach Theorem Applied to 11.2.1.** A characteristically modern tool for the discovery and proof of theorems about linear approximation is the Hahn-Banach theorem, which is described briefly in I, B.5. We propose to indicate here how this theorem may be used to prove 11.2.1; it is equally useful in connection with the analogous problems mentioned in 11.2.3 and 12.11.4.

It must be admitted that its application to the proof of 11.2.1 does not appear to be particularly economical, and it must be stressed that the great merit of the theorem lies rather in the range of problems to which it provides a useful common approach (see [E], Chapter 2). No account of the methods of modern analysis can afford to ignore it.

The notation being as in 11.2.1, let us face anew the problem of showing that $f \in \mathbf{I}$ whenever $Z_f \supset Z_{\mathbf{I}}$. Since **I** is a closed linear subspace of **E**, the Hahn-Banach theorem (specifically I, B.5.2) affirms that to do this it suffices (and is obviously necessary) to prove that, if F is any continuous linear functional on **E**, and if

$$F(g) = 0 \qquad \text{for all } g \in \mathbf{I}, \tag{11.2.1}$$

and

$$Z_f \supset Z_{\mathbf{I}}, \tag{11.2.2}$$

then

$$F(f) = 0. \tag{11.2.3}$$

Now, since **I** is invariant, (11.2.1) entails that

$$F(T_a g) = 0 \qquad \text{for all } g \in \mathbf{I} \text{ and all } a. \tag{11.2.4}$$

This suggests that we look at the function ϕ_g defined by

$$\phi_g(a) = F(T_a g). \tag{11.2.5}$$

Since F is continuous on **E**, while $a \to T_a g$ is continuous from R into **E** (see 2.2.4), ϕ_g is a continuous function. The reader will also observe for future use the fact that ϕ_g depends linearly and continuously on the variable $g \in \mathbf{E}$:

$$\|\phi_g\|_\infty \leqslant \|F\| \cdot \|g\|_{\mathbf{E}}.$$

The combination of these last remarks with a simple argument involving Riemann sums permits the computation of the Fourier coefficients of ϕ_g.

Thus (using an obvious notation),

$$\hat{\phi}_g(n) = \lim \frac{1}{2\pi} \sum \phi_g(a_k)e^{-ina_k} \Delta a_k$$

$$= \lim F\left(\frac{1}{2\pi} \sum T_{a_k}g \cdot e^{-ina_k} \Delta a_k\right),$$

by linearity of F, which in turn is equal to

$$F\left(\lim \frac{1}{2\pi} \sum T_{a_k}g \cdot e^{-ina_k} \Delta a_k\right)$$

on account of continuity of F. Now, if g is continuous, it is easy to check that the limit appearing in the last expression displayed is none other than the function

$$x \to \frac{1}{2\pi} \int g(x - a)e^{-ina} \, da,$$

which is, by virtue of the basic properties of the invariant integral recounted in 2.2.2, the function $\hat{g}(-n) \cdot e_{-n}$. Accordingly, the formula

$$\hat{\phi}_g(n) = \hat{g}(-n) \cdot F(e_{-n}) \tag{11.2.6}$$

is established for continuous $g \in \mathbf{E}$. However, for a fixed $n \in Z$, each side of (11.2.6) is a continuous linear functional of $g \in \mathbf{E}$; since the continuous functions are everywhere dense in $\mathbf{E}$ (a corollary of 2.4.4), (11.2.6) must hold for all $g \in \mathbf{E}$. The reader is urged to verify carefully all the steps in this computation of $\hat{\phi}_g$.

In view of (11.2.6), (11.2.4) entails that $F(e_n) = 0$ whenever $g \in \mathbf{I}$ and $\hat{g}(n) \neq 0$. Therefore

$$F(e_n) = 0 \qquad \text{for all } n \in Z \backslash Z_{\mathbf{I}}. \tag{11.2.7}$$

On the other hand, for any $f \in \mathbf{E}$ we have from 6.1.1

$$f = \lim_{N \to \infty} \sum_{|n| \leqslant N} \left(1 - \frac{|n|}{N + 1}\right)\hat{f}(n)e_n.$$

So, by linearity and continuity of F,

$$F(f) = \lim_{N \to \infty} \sum_{|n| \leqslant N} \left(1 - \frac{|n|}{N + 1}\right)\hat{f}(n)F(e_n). \tag{11.2.8}$$

Finally, by (11.2.2) and (11.2.7),

$$\hat{f}(n)F(e_n) = 0 \qquad \text{for all } n \in Z,$$

so that (11.2.3) follows from (11.2.8). This completes the proof.

Remarks. The computation of the Fourier coefficients of ϕ_g could be made to proceed more gracefully by appealing to the results of Chapter 12 and Appendix I, C.1 concerning the analytic representation of continuous linear functionals F on **E**. However, we have preferred at this stage to sacrifice grace in favor of more elementary arguments.

11.2.3. **Closure of Translations Theorems.** The writer knows of no very significant applications of 11.2.1 to problems of concrete analysis, though it has its own interest as a structure theorem, albeit a simple one. However, it and certain corollaries one can deduce from it have analogues for other groups which are at once deeper and productive of genuinely significant results in concrete analysis. We propose to mention these analogues, devoting this subsection to so-called "closure of translations" theorems, and the next to some consequences of a Tauberian nature (see 5.3.5).

The position is that 11.2.1 and its derivatives, pertaining to the group T, are simple prototypes of bigger and better things which owe their significance to their applicability to noncompact groups.

When one contemplates replacing the compact group T by a noncompact (locally compact Abelian) group G, it is difficult to repress the hope that an analogue of the case $\mathbf{E} = \mathbf{L}^1$ of 11.2.1 lurks around the corner and awaits discovery. There is little difficulty in framing a plausible analogue, and this plausible analogue turns out to be "approximately true," or to be "true in spirit but false in detail." (Concerning $\mathbf{L}^1(G)$ for a general group, see, for example, [R], Chapter 1; [HR], Section 20; [E], Section 4.19; [Bo]; [Bo$_1$], Chapitre 2.)

The simplest choice for a noncompact group G would undoubtedly be the group Z. Despite this, the description immediately following is expressed in terms of the groups R^m (R the additive group of real numbers with its usual topology and m a natural number). One reason for this choice is that R^m is more typical of noncompact groups than is Z. Another reason is that the original "closure of translations" theorem of Wiener (see [Wi], pp. 99–100), which was the beginning of almost everything in this field, applies to the group R. The analogous problems for the technically somewhat simpler group Z will receive further attention in 11.2.5 and 12.11.4.

For $f \in \mathbf{L}^1(R^m)$, the Fourier transform of f is the function on R^m defined by

$$\hat{f}(\xi) = \int \cdots \int_{R^m} f(x_1, \cdots, x_m) e^{-2\pi i(\xi_1 x_1 + \cdots + \xi_m x_m)}\, dx_1 \cdots dx_m$$

for $\xi = (\xi_1, \cdots, \xi_m) \in R^m$; Z_f is defined to be the set of zeros of $\hat{f}$; and, for any ideal $\mathbf{I}$ in $\mathbf{L}^1(R^m)$, $Z_{\mathbf{I}}$ is defined to be the intersection of the sets Z_f when f ranges over $\mathbf{I}$. (A brief treatment of Fourier transforms of functions in $\mathbf{L}^1(R)$ and $\mathbf{L}^2(R)$ appears in Chapters 9 and 19 of [R$_1$]; see also [Wi] and [Ti], and the references cited therein.)

The Wiener *closure of translations theorem* for R^m asserts that an ideal $\mathbf{I}$ in $\mathbf{L}^1(R^m)$ is everywhere dense in $\mathbf{L}^1(R^m)$ if (and only if) $Z_{\mathbf{I}} = \varnothing$. This is a perfect analogue of the corresponding special case of 11.2.1, and is indeed encouraging.

For quite a while it remained tantalizingly in doubt whether a general closed ideal $\mathbf{I}$ in $\mathbf{L}^1(R^m)$ necessarily contains every $f \in \mathbf{L}^1(R^m)$ such that $Z_f \supset Z_{\mathbf{I}}$. The first example showing that this was *not* always the case was given by Laurent Schwartz [1] in 1948 and applied to R^m with $m \geqslant 3$; see also Reiter [1] and 12.11.5. Another decade was to elapse before similar examples pertaining first to R, and then to any noncompact G, were produced by Malliavin [1] in 1959.

Despite this disappointment, it turns out that if $Z_{\mathbf{I}}$ is topologically simple enough, then $\mathbf{I}$ does indeed contain every $f \in \mathbf{L}^1(R^m)$ for which $Z_f \supset Z_{\mathbf{I}}$; and that the conclusion stands, whatever $Z_{\mathbf{I}}$, if in addition $\hat{f}$ is subject to smoothness conditions. Results of this type permit the reader to judge for himself to what extent the analogue of 11.2.1 (for $\mathbf{E} = \mathbf{L}^1$) may be claimed to be "approximately true." See [HR], (39.24); [Re], p. 28; [Kz], p. 225; [R], 7.2.4; MR **37** # 6694; **40** # 6491; **46** ## 9650, 9652, **49** # 9542; **53** # 14025; **54** ## 10980, 13464.

A set $S \subset R^m$ having the property that

$$f \in \mathbf{L}^1(R^m), \quad Z_f \supset Z_{\mathbf{I}} \;\Rightarrow\; f \in \mathbf{I}$$

for every closed ideal $\mathbf{I}$ in $\mathbf{L}^1(R^m)$ for which $Z_{\mathbf{I}} = S$, is termed a *spectral* (or *harmonic*) *synthesis set* in R^m; Rudin ([R], p. 158) refers to them more briefly as *S-sets*. It is known that S is a spectral synthesis set in this sense if and only if there is but one closed ideal $\mathbf{I}$ in $\mathbf{L}^1(R^m)$ satisfying $Z_{\mathbf{I}} = S$.

Malliavin's result cited above asserts precisely that there exist closed subsets of R^m which are *not* spectral synthesis sets. On the other hand, the opening statement in the last paragraph but one amounts to saying that conditions of topological simplicity are known which ensure that a given closed set S is a spectral synthesis set; compare Exercise 12.52.

Malliavin's result cited above has given rise to many extensions, improvements and simplifications. For some (if not all) of the details, the reader should consult Malliavin [1], [2]; [R], Chapter 7; [KS], Chapitre IX; [Kz], pp. 229 ff; [HR], §42; de Leeuw and Herz [1]; MR **31** # 2567; **39** # 1977; Exercise 12.53 below. At this point we remark merely that Malliavin's original construction has been simplified by Kahane and Katznelson [2] and Richards [1]; and that Varopoulos [1], [2] introduced an entirely original (tensor product) approach to spectral synthesis problems in Banach algebras; see MR **41** # 830 and the remarks in 11.4.18(4) below.

As has been indicated, strictly analogous problems arise when attention is transferred from $\mathbf{L}^1(R^m)$ to $\ell^1(Z)$; concerning this particular extension we shall have a little more to say in Subsections 11.2.5, 12.11.4, 12.11.5, and 12.11.6.

Mention must also be made of analogues for noncompact groups G of the remaining cases covered by 11.2.1, namely, the closure of translations theorems in $\mathbf{E} = \mathbf{L}^p$ $(1 < p < \infty)$ and $\mathbf{E} = \mathbf{C}$. The results for $\mathbf{L}^\infty(G)$ with its weak topology (see the opening remarks to this chapter) go hand in hand with those for $\mathbf{L}^1(G)$ already discussed. For $\mathbf{L}^2(R)$ a complete solution was given by Wiener ([Wi], p. 100), and this extends without trouble to $\mathbf{L}^2(G)$. In all other cases, that is, for values of p different from 1, 2, and ∞, the known results are less complete. While conditions are known which are sufficient to ensure that the linear combinations of the translates of a given $f \in \mathbf{L}^p(G)$ are everywhere dense in that space, and yet others are known which are necessary for this to happen, there remains a gap between the two types of conditions. All attacks on this problem are bedevilled by the preliminary task of devising and handling a tractable definition of the Fourier transform of a function belonging to an arbitrary space $\mathbf{L}^p(G)$. This may be done in terms of pseudomeasures and similar objects (the periodic prototypes of which are mentioned in Section 12.11; see especially 12.11.4). There is, alas, no connected account in book form, but see Herz's survey article [2], Gaudry [1], [3], Edwards [4] and the references there cited, and Warner [1]. (The case of the group Z is discussed briefly in 11.2.5.). See also MR **38** # 4904.

One striking fact, applying when G is noncompact and $1 < p < 2$, is that there exists a closed invariant subspace $\mathbf{V} \neq \{0\}$ in $\mathbf{L}^2(G)$ which contains no nonzero element of $\mathbf{L}^p(G)$; see MR **52** # 14849. See also MR **48** # 11915.

Finally, an even wider diversity obtains when one turns to analogues of the case $\mathbf{E} = \mathbf{C}$ of 11.2.1. This is due to the fact that there are, in relation to a noncompact G, several natural spaces of continuous functions which coalesce for compact groups but which otherwise are widely different. The following four contenders have received attention:

(1) the space $\mathbf{C}(G)$ of all continuous functions on G, with the topology of locally uniform convergence;

(2) the space $\mathbf{BC}(G)$ of bounded, continuous functions on G, with the topology of uniform convergence;

(3) the space $\mathbf{BUC}(G)$ of bounded, uniformly continuous functions on G, with the topology of uniform convergence;

(4) the space $\mathbf{C}_0(G)$ of continuous functions which tend to zero at infinity, with the topology of uniform convergence.

For $\mathbf{C}_0(G)$ fairly complete results are known. For the remaining three, results are hard to come by; in the case of $\mathbf{BC}(G)$ and $\mathbf{BUC}(G)$, more progress has been made concerning approximation relative to a weaker (the so-called "strict") topology, originally suggested by ideas of Beurling; see Edwards [5] and Harasymiv [1]. In the case of $\mathbf{C}(G)$, most attention has been paid to functions, the linear combinations of translates of which are *not* everywhere dense in $\mathbf{C}(G)$: these were introduced and studied (for $G = R$) by Laurent Schwartz [2] in 1947, who christened them *mean periodic functions*; see also [Kah_3]. Some of Schwartz's results have since been extended to more general groups by Ehrenpreis [1], Elliott [1], Gilbert [1], and others.

11.2.4. **About Tauberian Theorems.** We pass on to consider briefly some consequences of such closure of translations theorems as are typified by the case $Z_1 = \varnothing$ of 11.2.1 and the generalizations thereof mentioned in 11.2.3.

Let us begin with the group T. Suppose we take a subset A of T and a nonvoid collection Π of nonvoid subsets of A satisfying the following two conditions:

(1) the intersection of any two members of Π contains a member of Π (in M. Bourbaki's language, this signifies that Π is a *filter base* on A);

(2) if $a \in A$ and $P \in \Pi$, there exists $P' \in \Pi$ such that

$$P' \subset a + P \equiv \{a + x : x \in P\}.$$