McGRAW-HILL SERIES IN THE

GEOLOGICAL SCIENCES

Elastic Waves in Layered Media

W. MAURICE EWING

Director, Lamont Geological Observatory
Professor of Geology, Columbia University

WENCESLAS S. JARDETZKY

Associate Professor of Mechanics, Manhattan College
Research Associate, Lamont Geological Observatory

FRANK PRESS

Professor of Geophysics, California Institute of Technology

LAMONT GEOLOGICAL OBSERVATORY CONTRIBUTION No. 189

McGRAW-HILL BOOK COMPANY, INC.

NEW YORK TORONTO LONDON

1957

ELASTIC WAVES IN LAYERED MEDIA

Library of Congress Catalog Card Number 56-7558

PREFACE

This work is the outgrowth of a plan to make a uniform presentation of the investigations on earthquake seismology, underwater sound, and model seismology carried on by the group connected with Lamont Geological Observatory of Columbia University. The scope was subsequently enlarged to cover a particular selection of related problems. The methods and results of the theory of wave propagation in layered media are important in seismology, in geophysical prospecting, and in many problems of acoustics and electromagnetism.

Although the mathematical discussions of electromagnetic waves, water waves, and shock waves are very close to the methods used in this book, we had to reduce them to a few brief references. Many of the methods which have been used in seismological problems were originally developed in studies on electromagnetic waves. It is hoped that a systematic presentation of problems concerning elastic-wave propagation may now be useful in other fields.

The experimental viewpoint has, to a large extent, governed the selection of problems. For many years, research in seismology has been characterized by separation of the experimental and theoretical methods. The interplay of the two methods guided the research program which led to this book, and it has been retained whenever possible. Observations of surface waves from explosions and earthquakes, flexural waves in ice, and SOFAR sound propagation are a few examples of topics in which the theoretical and practical investigations benefited each other.

An effort was made to compile a comprehensive and systematic bibliography of the world literature for the main topics discussed. Few workers in this field could become familiar with all the past investigations, which are scattered in many journals.

We are very grateful to the Air Force Cambridge Research Center, the Bureau of Ships, and the Office of Naval Research for support of the program of research on elastic-wave propagation at the Lamont Geological Observatory. Peter Gottlieb, Dr. Samuel Katz, Dr. A. Laughton, Dr. Franklyn Levin, and Stefan Mueller kindly read the manuscript and made helpful suggestions.

Maurice Ewing
Wenceslas Jardetzky
Frank Press

LIST OF SYMBOLS

c_R	Velocity of Rayleigh waves
c	Phase velocity
E	Young's modulus
$e_{xx}, e_{xy}, \cdots, e_{zz}$	Strain components
$\mathfrak{e}$	Angle of emergence
f	Frequency or constant of gravitation
$\mathfrak{f}$	Angle of incidence for shear waves
$H_n^{(1)}, H_n^{(2)}$	Hankel functions of the order n
I_n	Modified Bessel function of the first kind of the order n
$\mathfrak{i}$	Angle of incidence
J_n	Bessel functions of the order n
k	Wave number
k	Coefficient of incompressibility
$\mathcal{K}_n$	Modified Bessel function of the second kind of the order n
l_0, l	Wave length
$p_{xx}, p_{xy}, \cdots, p_{zz}$	Stress components
$\mathcal{P}$	Principal value (of an integral)
p	Hydrostatic pressure
q, w	Displacement components in cylindrical coordinates
$\mathbf{s}(u, v, w)$	Displacement
T	Period
U	Group velocity
$\mathbf{v}(\bar{u}, \bar{v}, \bar{w})$	Velocity
X, Y, Z	Body forces
α	Compressional-wave velocity
β	Shear-wave velocity
γ	Parameter
ϵ	Phase shift
θ	Cubical dilatation or an angle
θ_{cr}	Critical angle
κ	Root of the Rayleigh equation or parameter
λ, μ	Lamé constants
ρ	Density
σ	Poisson's ratio or an angle
$\varphi, \boldsymbol{\psi}(\psi_1, \psi_2, \psi_3)$	Displacement potentials
$\bar{\varphi}$	Velocity potential
$\boldsymbol{\Omega}(\Omega_x, \Omega_y, \Omega_z)$	Rotation
ω	Angular frequency

CONTENTS

CHAPTER 1

FUNDAMENTAL EQUATIONS AND SOLUTIONS

1–1. Equations of Motion. The problems we shall consider concern the propagation of elastic disturbances in layered media, each layer being continuous, isotropic, and of constant thickness. We begin with a brief outline of the theory of motion in elastic media and a derivation of the equations of motion. A more detailed treatment may be found in reference books, e.g., Sommerfeld [57].†

When a deformable body undergoes a change in configuration due to the application of a system of forces, the body is said to be strained. Within the body, any point P with space-fixed rectangular coordinates (x, y, z) is then displaced to a new position, the components of displacement being, respectively, u, v, w. If Q is a neighboring point $(x + \Delta x, y + \Delta y, z + \Delta z)$, its displacement components can be given by a Taylor expansion in the form

$$\begin{aligned} &u + \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \frac{\partial u}{\partial z}\,\Delta z + \cdots \\ &v + \frac{\partial v}{\partial x}\,\Delta x + \frac{\partial v}{\partial y}\,\Delta y + \frac{\partial v}{\partial z}\,\Delta z + \cdots \\ &w + \frac{\partial w}{\partial x}\,\Delta x + \frac{\partial w}{\partial y}\,\Delta y + \frac{\partial w}{\partial z}\,\Delta z + \cdots \end{aligned} \tag{1–1}$$

For the small strains associated with elastic waves, higher-order terms can be neglected. Then, introducing the expressions

$$\Omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{1–2}$$

and others obtained by the cyclic change of letters x, y, z and u, v, w, respectively, we may write the displacement components (1–1) in the form

$$\begin{aligned} &u + (\Omega_y\Delta z - \Omega_z\Delta y) + (e_{xx}\Delta x + e_{xy}\Delta y + e_{xz}\Delta z) \\ &v + (\Omega_z\Delta x - \Omega_x\Delta z) + (e_{yx}\Delta x + e_{yy}\Delta y + e_{yz}\Delta z) \\ &w + (\Omega_x\Delta y - \Omega_y\Delta x) + (e_{zx}\Delta x + e_{zy}\Delta y + e_{zz}\Delta z) \end{aligned} \tag{1–3}$$

†Numerals in brackets in the text correspond to the numbered references at the end of the chapter.

The first terms of these expressions are the components of displacement of the point P. It can be shown that the terms in the first parentheses correspond to a pure rotation of a volume element and that the terms in the second parentheses are associated with deformation or strain of the element. The array

$$\begin{matrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{matrix} \tag{1–4}$$

represents the symmetrical strain tensor at P, since $e_{xy} = e_{yx} \cdots$. The three components

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{yy} = \frac{\partial v}{\partial y} \qquad e_{zz} = \frac{\partial w}{\partial z}$$

represent simple extensions parallel to the x, y, z axes, and the other three expressions e_{xy}, e_{yz}, e_{zx} are the shear components of strain, which may be shown to be equal to half the angular changes in the xy, yz, zx planes, respectively, of an originally orthogonal volume element. It is also shown in the theory of elasticity that there is a particular set of orthogonal axes through P for which the shear components of strain vanish. These axes are known as the principal axes of strain. The corresponding values of e_{xx}, e_{yy}, e_{zz} are the principal extensions which completely determine the deformation at P. Thus the deformation at any point may be specified by three mutually perpendicular extensions. It is also known that the sum $e_{xx} + e_{yy} + e_{zz}$ is independent of the choice of the orthogonal coordinate system.

The cubical dilatation θ, defined as the limit approached by the ratio of increase in volume to the initial volume when the dimensions Δx, Δy, Δz approach zero, is

$$\lim \frac{(\Delta x + e_{xx}\,\Delta x)(\Delta y + e_{yy}\,\Delta y)(\Delta z + e_{zz}\,\Delta z) - \Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z}$$

or

$$\theta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{1–5}$$

neglecting higher-order terms. Although the principal extensions e_{xx}, e_{yy}, e_{zz} are used in the derivation of (1–5), the result holds for any cartesian system because of the invariance of the sum.

Forces acting on an element of area ΔS separating two small portions of a body are, in general, equivalent to a resultant force or traction $\mathbf{R}$ upon the element and a couple $\mathbf{C}$ (Fig. 1–1). As ΔS goes to zero, the limit of the ratio of traction upon ΔS to the area ΔS is finite and defines the stress. The ratio of the couple to ΔS, involving an additional dimension of length, may be neglected. For a complete specification of the stress at P, it is necessary to give the traction at P acting upon all planes passing through the point. However, all these tractions may be reduced to com-

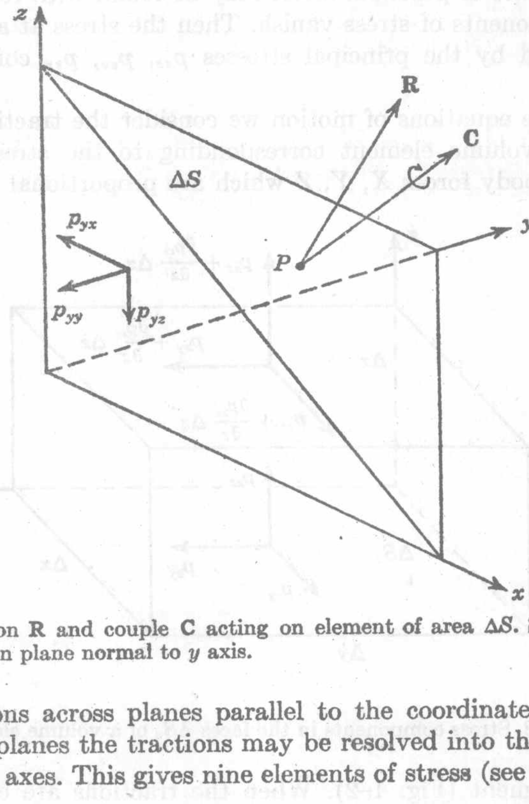

FIG. 1–1. Traction **R** and couple **C** acting on element of area ΔS. Stress components p_{yy}, p_{yz}, and p_{yx} in plane normal to y axis.

ponent tractions across planes parallel to the coordinate planes. Across each of these planes the tractions may be resolved into three components parallel to the axes. This gives nine elements of stress (see Fig. 1–1)

$$\begin{matrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{matrix} \qquad (1\text{–}6)$$

where the first subscripts represent a coordinate axis normal to a given plane and the second subscripts represent the axis to which the traction is parallel. The array (1–6) is a symmetrical tensor. This may be proved by considering the equilibrium of a small volume element within the medium with sides of length Δx, Δy, Δz, parallel to the x, y, z axes. Moments about axes through the center of mass arise from tractions corresponding to stresses p_{xy}, p_{yz}, ... Moments of normal stresses vanish, since the corresponding forces intersect the axes through the center of mass of the infinitesimal element and moments of body forces are small quantities of higher order than those of stresses. The equilibrium conditions require, therefore, that the shear components of stress be equal in pairs, $p_{xy} = p_{yx}$, etc. As was the case for the shear components of strain, three mutually perpendicular axes of principal stress may be found with respect to which the shear components of stress vanish. Then the stress at a point is completely specified by the principal stresses p_{xx}, p_{yy}, p_{zz} corresponding to these axes.

To derive the equations of motion we consider the tractions across the surfaces of a volume element corresponding to the stress components (1–6) and the body forces X, Y, Z which are proportional to the mass in

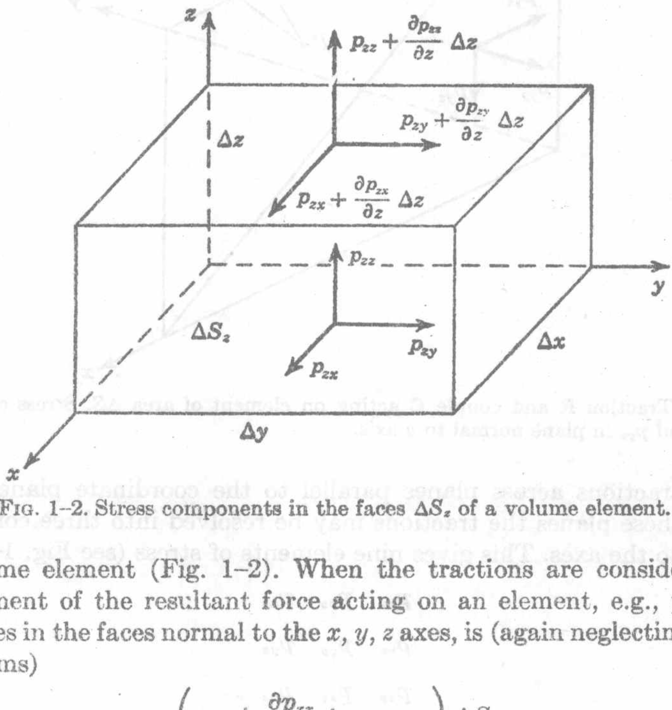

FIG. 1–2. Stress components in the faces ΔS_z of a volume element.

the volume element (Fig. 1–2). When the tractions are considered, the x component of the resultant force acting on an element, e.g., produced by stresses in the faces normal to the x, y, z axes, is (again neglecting higher-order terms)

$$\left(p_{xx} + \frac{\partial p_{xx}}{\partial x} \Delta x - p_{xx}\right) \Delta S_x$$

$$\left(p_{yx} + \frac{\partial p_{yx}}{\partial y} \Delta y - p_{yx}\right) \Delta S_y$$

$$\left(p_{zx} + \frac{\partial p_{zx}}{\partial z} \Delta z - p_{zx}\right) \Delta S_z$$

where ΔS_x, ΔS_y, ΔS_z are the areas of the faces normal to the x, y, z axes, respectively. It follows that the x component of force resulting from all the tractions is given by the three terms

$$\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}\right) \Delta x \, \Delta y \, \Delta z$$

The equations of motion are obtained by adding all the forces and the inertia terms $-\rho \, d^2u/dt^2 \, \Delta x \Delta y \Delta z, \cdots$, for each component:

$$\rho \frac{d^2u}{dt^2} = \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z}$$

$$\rho \frac{d^2v}{dt^2} = \rho Y + \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} \tag{1-7}$$

$$\rho \frac{d^2w}{dt^2} = \rho Z + \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z}$$

In these expressions, ρ is the density of the medium.

The Equation of Continuity. This equation expresses the condition that the mass of a given portion of matter is conserved. The total outflow of mass from the elementary volume $\Delta\tau$ during the time Δt is div $\rho\mathbf{v} \, \Delta\tau \, \Delta t$, where $\mathbf{v}$ is the velocity, whose components parallel to the x, y, z axes are $\bar{u}$, $\bar{v}$, $\bar{w}$. The loss of mass during the same time is $-(\partial\rho/\partial t) \, \Delta\tau \, \Delta t$. Equating these last two expressions gives

$$\frac{\partial \rho}{\partial t} + \text{div } \rho\mathbf{v} = 0 \tag{1-8}$$

Another form of this equation is

$$\frac{d\rho}{dt} + \rho \text{ div } \mathbf{v} = 0 \tag{1-9}$$

where the operation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad} \tag{1-10}$$

represents the "total or material" rate of change following the motion and $\partial/\partial t$ is the local rate of change.

1-2. Elastic Media. In the generalized form of Hooke's law, it is assumed that each of the six components of stress is a linear function of all the components of strain, and in the general case 36 elastic constants appear in the stress-strain relations.

Isotropic Elastic Solid. On account of the symmetry associated with an isotropic body, the number of elastic constants degenerates to two, and the stress-strain relations may be written in the following manner, using

Lamé's constants λ and μ:

$$
\begin{aligned}
p_{xx} &= \lambda\theta + 2\mu \frac{\partial u}{\partial x} & p_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
p_{yy} &= \lambda\theta + 2\mu \frac{\partial v}{\partial y} & p_{yz} &= \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad (1\text{–}11) \\
p_{zz} &= \lambda\theta + 2\mu \frac{\partial w}{\partial z} & p_{zx} &= \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)
\end{aligned}
$$

We also could have written these equations using any two of the constants: Young's modulus E, Poisson's ratio σ, or the coefficient of incompressibility k. The relations between these elastic constants are given by the equations

$$
\begin{aligned}
\lambda &= \frac{\sigma E}{(1+\sigma)(1-2\sigma)} & \mu &= \frac{E}{2(1+\sigma)} \\
E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} & \sigma &= \frac{\lambda}{2(\lambda+\mu)} \qquad (1\text{–}12) \\
k &= \lambda + \tfrac{2}{3}\mu
\end{aligned}
$$

Using Eqs. (1–7) and (1–11), we can write the equations of motion in terms of displacements u, v, w of a point in an elastic solid:

$$
\begin{aligned}
\rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu)\frac{\partial \theta}{\partial x} + \mu\nabla^2 u + \rho X \\
\rho \frac{\partial^2 v}{\partial t^2} &= (\lambda + \mu)\frac{\partial \theta}{\partial y} + \mu\nabla^2 v + \rho Y \qquad (1\text{–}13) \\
\rho \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu)\frac{\partial \theta}{\partial z} + \mu\nabla^2 w + \rho Z
\end{aligned}
$$

We have replaced d^2/dt^2 by $\partial^2/\partial t^2$, since it follows from (1–10) that the difference between corresponding expressions involves second powers or products of components which are assumed to be small. By neglecting these products, we linearize our differential equations.

For many solids, λ and μ are nearly equal, and we will occasionally use the Poisson relation $\lambda = \mu$ as a simplification. This corresponds to $k = \frac{5}{3}\mu$ and $\sigma = \frac{1}{4}$.

For an incompressible medium, $\theta = \operatorname{div} \mathbf{v} = 0$ or, by Eq. (1–9), $d\rho/dt = 0$.

Ideal Fluid. If the rigidity μ vanishes, the medium is an ideal fluid. From (1–11) and (1–12) we find $p_{xx} = p_{yy} = p_{zz} = k\theta = -p$, where $-p$, the value of the remaining independent component of the stress tensor, is the hydrostatic or mean pressure. In liquids the incompressibility k is very large, whereas it has only moderate values for gases. If a liquid is incompressible, $k = \infty$ and $\sigma = 0.5$. The equations of small motion in an ideal fluid may be obtained from (1–13) with $\mu = 0$.

1–3. Imperfectly Elastic Media. We shall also be concerned with the damping of elastic waves resulting from imperfections in elasticity, particularly from "internal friction." (For a discussion, see Birch [9, pp. 88–91].) The effect of internal friction may be introduced into the equations of motion by replacing an elastic constant such as μ by $\mu + \mu' \, \partial/\partial t$ in the equations of motion. This is equivalent to stating that stress is a linear function of both the strain and the time rate of change of strain. For simple harmonic motion, the time factor $e^{i\omega t}$ is used, and the effect of internal friction is introduced by replacing μ by the complex rigidity $\mu(1 + i/Q)$, where $1/Q = \omega\mu'/\mu$. In many cases Q may be treated as independent of frequency to a sufficiently good approximation but the more detailed discussion which this case requires is given in Sec. 5–6.

1–4. Boundary Conditions. If the medium to which the equations of motion are applied is bounded, some special conditions must be added. These conditions express the behavior of stresses and displacement at the boundaries. At a free surface of a solid or liquid all stress components vanish. In the problems which follow it will be assumed that solid elastic media are welded together at the surface of contact, implying continuity of all stress and displacement components across the boundary. At a solid-liquid interface slippage can occur, and continuity of normal stresses and displacements alone is required. Since the rigidity vanishes in the liquid, tangential stresses in the solid must vanish at the interface.

1–5. Reduction to Wave Equations. The equations of motion of a fluid [derived from (1–13) with $\mu = 0$ and therefore $\lambda = k$] can be simplified and reduced to one differential equation if a velocity potential $\bar{\varphi}$, defined as follows, exists:

$$\bar{u} = \frac{\partial \bar{\varphi}}{\partial x} \qquad \bar{v} = \frac{\partial \bar{\varphi}}{\partial y} \qquad \bar{w} = \frac{\partial \bar{\varphi}}{\partial z} \tag{1–14}$$

If the body forces are neglected, Eqs. (1–13) reduce to

$$\rho \frac{\partial \bar{u}}{\partial t} = k \frac{\partial \theta}{\partial x} \qquad \rho \frac{\partial \bar{v}}{\partial t} = k \frac{\partial \theta}{\partial y} \qquad \rho \frac{\partial \bar{w}}{\partial t} = k \frac{\partial \theta}{\partial z} \tag{1–15}$$

Now, writing $\alpha^2 = k/\rho$, we easily see from (1–14) and (1–15) that

$$\frac{\partial \bar{\varphi}}{\partial t} = \alpha^2 \theta + F(t) \tag{1–16}$$

and

$$\bar{\varphi} = \alpha^2 \int_0^t \theta \, dt \tag{1–17}$$

where the additive function of t is omitted, being without significance.

From the definition of mean pressure $p = -k\theta$ and (1–16) we have

$$p = -\rho \frac{\partial \bar{\varphi}}{\partial t} \tag{1–18}$$

Then, from (1–16) and (1–5) we obtain

$$\nabla^2 \bar{\varphi} = \frac{1}{\alpha^2} \frac{\partial^2 \bar{\varphi}}{\partial t^2} \tag{1–19}$$

in which small quantities of higher order have been neglected. This wave equation holds for small disturbances propagating in an ideal fluid with velocity α, under the assumptions mentioned above.

For displacements in a solid body, it is convenient to define a scalar potential φ and a vector potential $\boldsymbol{\psi}(\psi_1, \psi_2, \psi_3)$ as follows:

$$u = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z}$$

$$v = \frac{\partial \varphi}{\partial y} + \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} \tag{1–20}$$

$$w = \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y}$$

or, in vector form,

$$\mathbf{s}(u, v, w) = \operatorname{grad} \varphi + \operatorname{curl} \boldsymbol{\psi}(\psi_1, \psi_2, \psi_3) \tag{1–20$'$}$$

By the definition of θ as given by (1–5), we obtain

$$\theta = \nabla^2 \varphi \tag{1–21}$$

In general, the equations of motion (1–13) represent the propagation of a disturbance which involves both equivoluminal ($\theta = 0$) and irrotational ($\boldsymbol{\Omega} = 0$) motion, where $\theta = \operatorname{div} \mathbf{s}(u, v, w)$ and $\boldsymbol{\Omega} = \frac{1}{2} \operatorname{curl} \mathbf{s}$ [see Eqs. (1–2)]. However, by introduction of the potentials φ and ψ_i, separate wave equations are obtained for these two types of motion. Assuming that the body forces may be neglected, we can write the first of Eqs. (1–13) in the form

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial^2 \varphi}{\partial t^2}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial^2 \psi_3}{\partial t^2}\right) - \frac{\partial}{\partial z}\left(\rho \frac{\partial^2 \psi_2}{\partial t^2}\right)$$

$$= (\lambda + \mu) \frac{\partial}{\partial x} \nabla^2 \varphi + \mu \frac{\partial}{\partial x} \nabla^2 \varphi + \mu \frac{\partial}{\partial y} \nabla^2 \psi_3 - \mu \frac{\partial}{\partial z} \nabla^2 \psi_2$$

It is easy to see that this equation and the two others from (1–13) written in a similar form will be satisfied if the functions φ and ψ_i are solutions of the equations

$$\nabla^2 \varphi = \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} \qquad \nabla^2 \psi_i = \frac{1}{\beta^2} \frac{\partial^2 \psi_i}{\partial t^2} \qquad i = 1, 2, 3 \tag{1–22}$$