

McGRAW·HILL SERIES IN THE
**GEOLOGICAL
SCIENCES**

Elastic Waves in Layered Media

W. MAURICE EWING

*Director, Lamont Geological Observatory
Professor of Geology, Columbia University*

WENCESLAS S. JARDETZKY

*Associate Professor of Mechanics, Manhattan College
Research Associate, Lamont Geological Observatory*

FRANK PRESS

Professor of Geophysics, California Institute of Technology

LAMONT GEOLOGICAL OBSERVATORY CONTRIBUTION No. 189

McGRAW-HILL BOOK COMPANY, INC.

NEW YORK TORONTO LONDON

1957

ELASTIC WAVES IN LAYERED MEDIA

Copyright © 1957 by the McGraw-Hill Book Company, Inc. Printed in the United States of America. All rights reserved. This book, or parts thereof, may not be reproduced in any form without permission of the publishers.

Library of Congress Catalog Card Number 56-7558

PREFACE

This work is the outgrowth of a plan to make a uniform presentation of the investigations on earthquake seismology, underwater sound, and model seismology carried on by the group connected with Lamont Geological Observatory of Columbia University. The scope was subsequently enlarged to cover a particular selection of related problems. The methods and results of the theory of wave propagation in layered media are important in seismology, in geophysical prospecting, and in many problems of acoustics and electromagnetism.

Although the mathematical discussions of electromagnetic waves, water waves, and shock waves are very close to the methods used in this book, we had to reduce them to a few brief references. Many of the methods which have been used in seismological problems were originally developed in studies on electromagnetic waves. It is hoped that a systematic presentation of problems concerning elastic-wave propagation may now be useful in other fields.

The experimental viewpoint has, to a large extent, governed the selection of problems. For many years, research in seismology has been characterized by separation of the experimental and theoretical methods. The interplay of the two methods guided the research program which led to this book, and it has been retained whenever possible. Observations of surface waves from explosions and earthquakes, flexural waves in ice, and SOFAR sound propagation are a few examples of topics in which the theoretical and practical investigations benefited each other.

An effort was made to compile a comprehensive and systematic bibliography of the world literature for the main topics discussed. Few workers in this field could become familiar with all the past investigations, which are scattered in many journals.

We are very grateful to the Air Force Cambridge Research Center, the Bureau of Ships, and the Office of Naval Research for support of the program of research on elastic-wave propagation at the Lamont Geological Observatory. Peter Gottlieb, Dr. Samuel Katz, Dr. A. Laughton, Dr. Franklyn Levin, and Stefan Mueller kindly read the manuscript and made helpful suggestions.

MAURICE EWING
WENCESLAS JARDETZKY
FRANK PRESS

LIST OF SYMBOLS

c_R	Velocity of Rayleigh waves
c	Phase velocity
E	Young's modulus
$e_{xx}, e_{xy}, \dots, e_{zz}$	Strain components
e	Angle of emergence
f	Frequency or constant of gravitation
f	Angle of incidence for shear waves
$H_n^{(1)}, H_n^{(2)}$	Hankel functions of the order n
I_n	Modified Bessel function of the first kind of the order n
i	Angle of incidence
J_n	Bessel functions of the order n
k	Wave number
k	Coefficient of incompressibility
\mathcal{K}_n	Modified Bessel function of the second kind of the order n
l_0, l	Wave length
$p_{xx}, p_{xy}, \dots, p_{zz}$	Stress components
\mathcal{P}	Principal value (of an integral)
p	Hydrostatic pressure
q, w	Displacement components in cylindrical coordinates
$s(u, v, w)$	Displacement
T	Period
U	Group velocity
$\mathbf{v}(\bar{u}, \bar{v}, \bar{w})$	Velocity
X, Y, Z	Body forces
α	Compressional-wave velocity
β	Shear-wave velocity
γ	Parameter
ϵ	Phase shift
θ	Cubical dilatation or an angle
θ_c	Critical angle
κ	Root of the Rayleigh equation or parameter
λ, μ	Lamé constants
ρ	Density
σ	Poisson's ratio or an angle
$\varphi, \psi(\psi_1, \psi_2, \psi_3)$	Displacement potentials
$\tilde{\varphi}$	Velocity potential
$\Omega(\Omega_x, \Omega_y, \Omega_z)$	Rotation
ω	Angular frequency

CONTENTS

<i>Preface</i>	v
<i>List of Symbols</i>	xi
1. FUNDAMENTAL EQUATIONS AND SOLUTIONS	1
1-1. Equations of Motion	1
The Equation of Continuity	5
1-2. Elastic Media	5
Isotropic Elastic Solid	5
Ideal Fluid	6
1-3. Imperfectly Elastic Media	7
1-4. Boundary Conditions	7
1-5. Reduction to Wave Equations	7
1-6. Solutions of the Wave Equation	10
Plane Waves	10
Spherical Waves	12
Poisson and Kirchhoff Solutions	16
General Solution of Wave Equation	17
Some Special Forms Used	18
References	20
2. HOMOGENEOUS AND ISOTROPIC HALF SPACE	24
2-1. Reflection of Plane Waves at a Free Surface	24
Incident <i>P</i> Waves	26
Incident <i>S</i> Waves	28
Partition of Reflected Energy	28
Reflection at Critical Angles	29
2-2. Free Rayleigh Waves	31
2-3. Integral Solutions for a Line Source	34
Surface Source	34
Internal Source	36
2-4. Integral Solutions for a Point Source	38
Surface Source	38
Internal Source	42
2-5. Evaluation of Integral Solutions	44
Application of Contour Integration	44
Residues	47
Branch Line Integrals: Line Source	49
Application of the Method of Steepest Descent	59
2-6. Generalization for an Arbitrary Time Variation	61
2-7. Other Investigations	64

2-8. Traveling Disturbance	67
2-9. Experimental Study of Lamb's Problem	69
References	71
 3. TWO SEMI-INFINITE MEDIA IN CONTACT	 74
3-1. Reflection and Refraction of Plane Waves at an Interface	74
Rigid Boundary	74
General Equations	76
Liquid-Liquid Interface	78
Liquid-Solid Interface	79
Solid-Solid Interface	83
3-2. Reflection of a Pulse Incident beyond the Critical Angle	90
3-3. Propagation in Two Semi-infinite Media: Point Source	93
Two Liquids	94
Fluid and Solid Half Spaces	105
Two Solids	107
Stoneley Waves	111
3-4. Further Remarks on Waves Generated at an Interface	113
3-5. Other Investigations	115
References	121
 4. A LAYERED HALF SPACE	 124
4-1. General Equations for an n -Layered Elastic Half Space	124
4-2. Two-layered Liquid Half Space	126
Discussion of Solutions	137
Generalization for a Pulse	142
4-3. Three-layered Liquid Half Space	151
4-4. Liquid Layer on a Solid Bottom	156
Compressional-wave Source in the Solid Substratum	157
Suboceanic Rayleigh Waves: First Mode	166
Compressional-wave Source in the Liquid Layer	174
Leaking Modes	184
Some Aspects of Microseisms	185
4-5. Solid Layer over Solid Half Space	189
Rayleigh Waves: General Discussion	189
Propagation of Rayleigh Waves across Continents	196
Ground Roll	200
Theoretical Rayleigh-wave Dispersion Curves	204
Love Waves: General Discussion	205
Love Waves across Continents	213
Love Waves across Oceans	216
Lg and Rg Waves	219
Other Investigations	222
4-6. Three-layered Half Space	224
Oceanic Rayleigh Waves with Layered Substratum	225
Love Waves	227
4-7. Air-coupled Rayleigh Waves	230
Air-coupled Ground Roll	236
4-8. Remarks Concerning the Problem of an n -Layered Half Space	238
References	245

5. THE EFFECTS OF GRAVITY, CURVATURE, AND VISCOSITY	255
5-1. Gravity Terms in General Equations	255
5-2. Effect of Gravity on Surface Waves	257
Rayleigh Waves: Incompressible Half Space	257
Gravitating and Compressible Liquid Layer over Solid Half Space	260
5-3. Effect of Curvature on Surface Waves	263
Cylindrical Curvature	263
Spherical Curvature	265
5-4. General Solutions for a Spherical Body	266
Wave Propagation in a Gravitating Compressible Planet	269
5-5. The Effect of Internal Friction	272
Voigt Solid	272
Maxwell Solid	276
Internal Friction in Earth Materials	277
References	278
6. PLATES AND CYLINDERS	281
6-1. Plate in a Vacuum	281
Symmetric Vibrations (M_1)	283
Antisymmetric Vibrations (M_2)	285
Interpretation in Terms of P and SV Waves	286
Impulsive Sources	286
Other Investigations	288
6-2. Plate in a Liquid	288
Symmetric Vibrations	292
Antisymmetric Vibrations	293
Other Investigations	293
6-3. Floating Ice Sheet	293
SH Waves	293
SV and P Waves	295
Crary Waves	299
Flexural Waves from an Impulsive Source	301
6-4. Cylindrical Rod in a Vacuum	305
Longitudinal Vibrations	306
Torsional and Flexural Vibrations	311
6-5. Cylindrical Rod in a Liquid	314
6-6. Cylindrical Hole in an Infinite Solid	314
6-7. Liquid Cylinder in an Elastic Medium	317
6-8. Cylindrical Tube	319
References	323
7. WAVE PROPAGATION IN MEDIA WITH VARIABLE VELOCITY	328
7-1. Wave Propagation in Heterogeneous Isotropic Media	328
7-2. Sound Propagation in a Fluid Half Space	330
SOFAR Propagation	335
The T Phase	341
7-3. Love Waves in Heterogeneous Isotropic Media	341
Homogeneous Layer over Heterogeneous Half Space: Matuzawa's Case	343
Satô's Case	344
Jeffreys' Case	346
Meissner's Case	347

CONTENTS

vii

7-4. Rayleigh Waves in Heterogeneous Isotropic Media	349
Mantle Rayleigh Waves	353
7-5. Aeolotropic and Other Media	358
References	360
APPENDIX A. METHOD OF STEEPEST DESCENT	365
References	369
APPENDIX B. RAYLEIGH'S PRINCIPLE	370
References	374
Index	375

CHAPTER 1

FUNDAMENTAL EQUATIONS AND SOLUTIONS

1-1. Equations of Motion. The problems we shall consider concern the propagation of elastic disturbances in layered media, each layer being continuous, isotropic, and of constant thickness. We begin with a brief outline of the theory of motion in elastic media and a derivation of the equations of motion. A more detailed treatment may be found in reference books, e.g., Sommerfeld [57].†

When a deformable body undergoes a change in configuration due to the application of a system of forces, the body is said to be strained. Within the body, any point P with space-fixed rectangular coordinates (x, y, z) is then displaced to a new position, the components of displacement being, respectively, u, v, w . If Q is a neighboring point $(x + \Delta x, y + \Delta y, z + \Delta z)$, its displacement components can be given by a Taylor expansion in the form

$$\begin{aligned} u + \frac{\partial u}{\partial x} \Delta x + \frac{\partial u}{\partial y} \Delta y + \frac{\partial u}{\partial z} \Delta z + \dots \\ v + \frac{\partial v}{\partial x} \Delta x + \frac{\partial v}{\partial y} \Delta y + \frac{\partial v}{\partial z} \Delta z + \dots \\ w + \frac{\partial w}{\partial x} \Delta x + \frac{\partial w}{\partial y} \Delta y + \frac{\partial w}{\partial z} \Delta z + \dots \end{aligned} \quad (1-1)$$

For the small strains associated with elastic waves, higher-order terms can be neglected. Then, introducing the expressions

$$\Omega_x = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \quad e_{xy} = \frac{1}{2} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \quad (1-2)$$

and others obtained by the cyclic change of letters x, y, z and u, v, w , respectively, we may write the displacement components (1-1) in the form

$$\begin{aligned} u + (\Omega_y \Delta z - \Omega_z \Delta y) + (e_{xx} \Delta x + e_{xy} \Delta y + e_{xz} \Delta z) \\ v + (\Omega_z \Delta x - \Omega_x \Delta z) + (e_{yx} \Delta x + e_{yy} \Delta y + e_{yz} \Delta z) \\ w + (\Omega_x \Delta y - \Omega_y \Delta x) + (e_{zx} \Delta x + e_{zy} \Delta y + e_{zz} \Delta z) \end{aligned} \quad (1-3)$$

†Numerals in brackets in the text correspond to the numbered references at the end of the chapter.

The first terms of these expressions are the components of displacement of the point P . It can be shown that the terms in the first parentheses correspond to a pure rotation of a volume element and that the terms in the second parentheses are associated with deformation or strain of the element. The array

$$\begin{array}{ccc} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{array} \quad (1-4)$$

represents the symmetrical strain tensor at P , since $e_{xy} = e_{yx} \dots$. The three components

$$e_{xx} = \frac{\partial u}{\partial x} \quad e_{yy} = \frac{\partial v}{\partial y} \quad e_{zz} = \frac{\partial w}{\partial z}$$

represent simple extensions parallel to the x, y, z axes, and the other three expressions e_{xy}, e_{yz}, e_{zx} are the shear components of strain, which may be shown to be equal to half the angular changes in the xy, yz, zx planes, respectively, of an originally orthogonal volume element. It is also shown in the theory of elasticity that there is a particular set of orthogonal axes through P for which the shear components of strain vanish. These axes are known as the principal axes of strain. The corresponding values of e_{xx}, e_{yy}, e_{zz} are the principal extensions which completely determine the deformation at P . Thus the deformation at any point may be specified by three mutually perpendicular extensions. It is also known that the sum $e_{xx} + e_{yy} + e_{zz}$ is independent of the choice of the orthogonal coordinate system.

The cubical dilatation θ , defined as the limit approached by the ratio of increase in volume to the initial volume when the dimensions $\Delta x, \Delta y, \Delta z$ approach zero, is

$$\lim \frac{(\Delta x + e_{xx} \Delta x)(\Delta y + e_{yy} \Delta y)(\Delta z + e_{zz} \Delta z) - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

or

$$\theta = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad (1-5)$$

neglecting higher-order terms. Although the principal extensions e_{xx}, e_{yy}, e_{zz} are used in the derivation of (1-5), the result holds for any cartesian system because of the invariance of the sum.

Forces acting on an element of area ΔS separating two small portions of a body are, in general, equivalent to a resultant force or traction \mathbf{R} upon the element and a couple \mathbf{C} (Fig. 1-1). As ΔS goes to zero, the limit of the ratio of traction upon ΔS to the area ΔS is finite and defines the

stress. The ratio of the couple to ΔS , involving an additional dimension of length, may be neglected. For a complete specification of the stress at P , it is necessary to give the traction at P acting upon all planes passing through the point. However, all these tractions may be reduced to com-

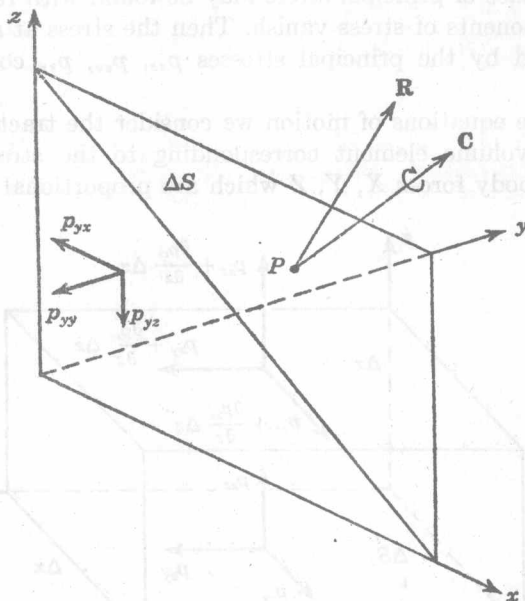


FIG. 1-1. Traction R and couple C acting on element of area ΔS . Stress components p_{yx} , p_{yy} , and p_{yz} in plane normal to y axis.

ponent tractions across planes parallel to the coordinate planes. Across each of these planes the tractions may be resolved into three components parallel to the axes. This gives nine elements of stress (see Fig. 1-1)

$$\begin{array}{ccc} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{array} \quad (1-6)$$

where the first subscripts represent a coordinate axis normal to a given plane and the second subscripts represent the axis to which the traction is parallel. The array (1-6) is a symmetrical tensor. This may be proved by considering the equilibrium of a small volume element within the medium with sides of length Δx , Δy , Δz , parallel to the x , y , z axes. Moments about axes through the center of mass arise from tractions corresponding to stresses p_{xy} , p_{yx} , \dots . Moments of normal stresses vanish, since the corresponding forces intersect the axes through the center of mass of the

infinitesimal element and moments of body forces are small quantities of higher order than those of stresses. The equilibrium conditions require, therefore, that the shear components of stress be equal in pairs, $p_{xy} = p_{yx}$, etc. As was the case for the shear components of strain, three mutually perpendicular axes of principal stress may be found with respect to which the shear components of stress vanish. Then the stress at a point is completely specified by the principal stresses p_{xx} , p_{yy} , p_{zz} corresponding to these axes.

To derive the equations of motion we consider the tractions across the surfaces of a volume element corresponding to the stress components (1-6) and the body forces X , Y , Z which are proportional to the mass in

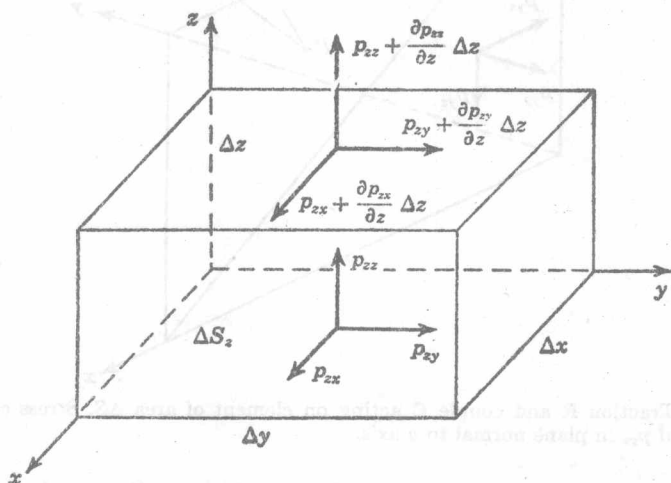


FIG. 1-2. Stress components in the faces ΔS_x of a volume element.

the volume element (Fig. 1-2). When the tractions are considered, the x component of the resultant force acting on an element, e.g., produced by stresses in the faces normal to the x , y , z axes, is (again neglecting higher-order terms)

$$\begin{aligned} & \left(p_{xx} + \frac{\partial p_{xx}}{\partial x} \Delta x - p_{xx} \right) \Delta S_x \\ & \left(p_{yx} + \frac{\partial p_{yx}}{\partial y} \Delta y - p_{yx} \right) \Delta S_y \\ & \left(p_{zx} + \frac{\partial p_{zx}}{\partial z} \Delta z - p_{zx} \right) \Delta S_z \end{aligned}$$

where ΔS_x , ΔS_y , ΔS_z are the areas of the faces normal to the x , y , z axes, respectively. It follows that the x component of force resulting from all

the tractions is given by the three terms

$$\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \right) \Delta x \Delta y \Delta z$$

The equations of motion are obtained by adding all the forces and the inertia terms $-\rho d^2u/dt^2 \Delta x \Delta y \Delta z, \dots$, for each component:

$$\begin{aligned} \rho \frac{d^2u}{dt^2} &= \rho X + \frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{yx}}{\partial y} + \frac{\partial p_{zx}}{\partial z} \\ \rho \frac{d^2v}{dt^2} &= \rho Y + \frac{\partial p_{xy}}{\partial x} + \frac{\partial p_{yy}}{\partial y} + \frac{\partial p_{zy}}{\partial z} \\ \rho \frac{d^2w}{dt^2} &= \rho Z + \frac{\partial p_{xz}}{\partial x} + \frac{\partial p_{yz}}{\partial y} + \frac{\partial p_{zz}}{\partial z} \end{aligned} \quad (1-7)$$

In these expressions, ρ is the density of the medium.

The Equation of Continuity. This equation expresses the condition that the mass of a given portion of matter is conserved. The total outflow of mass from the elementary volume $\Delta\tau$ during the time Δt is $\text{div } \rho \mathbf{v} \Delta\tau \Delta t$, where \mathbf{v} is the velocity, whose components parallel to the x, y, z axes are $\bar{u}, \bar{v}, \bar{w}$. The loss of mass during the same time is $-(\partial\rho/\partial t) \Delta\tau \Delta t$. Equating these last two expressions gives

$$\frac{\partial \rho}{\partial t} + \text{div } \rho \mathbf{v} = 0 \quad (1-8)$$

Another form of this equation is

$$\frac{d\rho}{dt} + \rho \text{div } \mathbf{v} = 0 \quad (1-9)$$

where the operation

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \text{grad} \quad (1-10)$$

represents the "total or material" rate of change following the motion and $\partial/\partial t$ is the local rate of change.

1-2. Elastic Media. In the generalized form of Hooke's law, it is assumed that each of the six components of stress is a linear function of all the components of strain, and in the general case 36 elastic constants appear in the stress-strain relations.

Isotropic Elastic Solid. On account of the symmetry associated with an isotropic body, the number of elastic constants degenerates to two, and the stress-strain relations may be written in the following manner, using

Lamé's constants λ and μ :

$$\begin{aligned} p_{xx} &= \lambda\theta + 2\mu \frac{\partial u}{\partial x} & p_{xy} &= \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\ p_{yy} &= \lambda\theta + 2\mu \frac{\partial v}{\partial y} & p_{yz} &= \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\ p_{zz} &= \lambda\theta + 2\mu \frac{\partial w}{\partial z} & p_{zx} &= \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \end{aligned} \quad (1-11)$$

We also could have written these equations using any two of the constants: Young's modulus E , Poisson's ratio σ , or the coefficient of incompressibility k . The relations between these elastic constants are given by the equations

$$\begin{aligned} \lambda &= \frac{\sigma E}{(1 + \sigma)(1 - 2\sigma)} & \mu &= \frac{E}{2(1 + \sigma)} \\ E &= \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} & \sigma &= \frac{\lambda}{2(\lambda + \mu)} \\ k &= \lambda + \frac{2}{3}\mu \end{aligned} \quad (1-12)$$

Using Eqs. (1-7) and (1-11), we can write the equations of motion in terms of displacements u, v, w of a point in an elastic solid:

$$\begin{aligned} \rho \frac{\partial^2 u}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial x} + \mu \nabla^2 u + \rho X \\ \rho \frac{\partial^2 v}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial y} + \mu \nabla^2 v + \rho Y \\ \rho \frac{\partial^2 w}{\partial t^2} &= (\lambda + \mu) \frac{\partial \theta}{\partial z} + \mu \nabla^2 w + \rho Z \end{aligned} \quad (1-13)$$

We have replaced d^2/dt^2 by $\partial^2/\partial t^2$, since it follows from (1-10) that the difference between corresponding expressions involves second powers or products of components which are assumed to be small. By neglecting these products, we linearize our differential equations.

For many solids, λ and μ are nearly equal, and we will occasionally use the Poisson relation $\lambda = \mu$ as a simplification. This corresponds to $k = \frac{5}{3}\mu$ and $\sigma = \frac{1}{4}$.

For an incompressible medium, $\theta = \text{div } \mathbf{v} = 0$ or, by Eq. (1-9), $d\rho/dt = 0$.

Ideal Fluid. If the rigidity μ vanishes, the medium is an ideal fluid. From (1-11) and (1-12) we find $p_{xx} = p_{yy} = p_{zz} = k\theta = -p$, where $-p$, the value of the remaining independent component of the stress tensor, is the hydrostatic or mean pressure. In liquids the incompressibility k is very large, whereas it has only moderate values for gases. If a liquid is

incompressible, $k = \infty$ and $\sigma = 0.5$. The equations of small motion in an ideal fluid may be obtained from (1-13) with $\mu = 0$.

1-3. Imperfectly Elastic Media. We shall also be concerned with the damping of elastic waves resulting from imperfections in elasticity, particularly from "internal friction." (For a discussion, see Birch [9, pp. 88-91].) The effect of internal friction may be introduced into the equations of motion by replacing an elastic constant such as μ by $\mu + \mu' \partial/\partial t$ in the equations of motion. This is equivalent to stating that stress is a linear function of both the strain and the time rate of change of strain. For simple harmonic motion, the time factor $e^{i\omega t}$ is used, and the effect of internal friction is introduced by replacing μ by the complex rigidity $\mu(1 + i/Q)$, where $1/Q = \omega\mu'/\mu$. In many cases Q may be treated as independent of frequency to a sufficiently good approximation but the more detailed discussion which this case requires is given in Sec. 5-6.

1-4. Boundary Conditions. If the medium to which the equations of motion are applied is bounded, some special conditions must be added. These conditions express the behavior of stresses and displacement at the boundaries. At a free surface of a solid or liquid all stress components vanish. In the problems which follow it will be assumed that solid elastic media are welded together at the surface of contact, implying continuity of all stress and displacement components across the boundary. At a solid-liquid interface slippage can occur, and continuity of normal stresses and displacements alone is required. Since the rigidity vanishes in the liquid, tangential stresses in the solid must vanish at the interface.

1-5. Reduction to Wave Equations. The equations of motion of a fluid [derived from (1-13) with $\mu = 0$ and therefore $\lambda = k$] can be simplified and reduced to one differential equation if a velocity potential $\bar{\varphi}$, defined as follows, exists:

$$\bar{u} = \frac{\partial \bar{\varphi}}{\partial x} \quad \bar{v} = \frac{\partial \bar{\varphi}}{\partial y} \quad \bar{w} = \frac{\partial \bar{\varphi}}{\partial z} \quad (1-14)$$

If the body forces are neglected, Eqs. (1-13) reduce to

$$\rho \frac{\partial \bar{u}}{\partial t} = k \frac{\partial \theta}{\partial x} \quad \rho \frac{\partial \bar{v}}{\partial t} = k \frac{\partial \theta}{\partial y} \quad \rho \frac{\partial \bar{w}}{\partial t} = k \frac{\partial \theta}{\partial z} \quad (1-15)$$

Now, writing $\alpha^2 = k/\rho$, we easily see from (1-14) and (1-15) that

$$\frac{\partial \bar{\varphi}}{\partial t} = \alpha^2 \theta + F(t) \quad (1-16)$$

and

$$\bar{\varphi} = \alpha^2 \int_0^t \theta dt \quad (1-17)$$

where the additive function of t is omitted, being without significance.

From the definition of mean pressure $p = -k\theta$ and (1-16) we have

$$p = -\rho \frac{\partial \bar{\varphi}}{\partial t} \quad (1-18)$$

Then, from (1-16) and (1-5) we obtain

$$\nabla^2 \bar{\varphi} = \frac{1}{\alpha^2} \frac{\partial^2 \bar{\varphi}}{\partial t^2} \quad (1-19)$$

in which small quantities of higher order have been neglected. This wave equation holds for small disturbances propagating in an ideal fluid with velocity α , under the assumptions mentioned above.

For displacements in a solid body, it is convenient to define a scalar potential φ and a vector potential $\psi(\psi_1, \psi_2, \psi_3)$ as follows:

$$\begin{aligned} u &= \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_3}{\partial y} - \frac{\partial \psi_2}{\partial z} \\ v &= \frac{\partial \varphi}{\partial y} + \frac{\partial \psi_1}{\partial z} - \frac{\partial \psi_3}{\partial x} \\ w &= \frac{\partial \varphi}{\partial z} + \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} \end{aligned} \quad (1-20)$$

or, in vector form,

$$\mathbf{s}(u, v, w) = \text{grad } \varphi + \text{curl } \psi(\psi_1, \psi_2, \psi_3) \quad (1-20')$$

By the definition of θ as given by (1-5), we obtain

$$\theta = \nabla^2 \varphi \quad (1-21)$$

In general, the equations of motion (1-13) represent the propagation of a disturbance which involves both equivoluminal ($\theta = 0$) and irrotational ($\Omega = 0$) motion, where $\theta = \text{div } \mathbf{s}(u, v, w)$ and $\Omega = \frac{1}{2} \text{curl } \mathbf{s}$ [see Eqs. (1-2)]. However, by introduction of the potentials φ and ψ_i , separate wave equations are obtained for these two types of motion. Assuming that the body forces may be neglected, we can write the first of Eqs. (1-13) in the form

$$\begin{aligned} \frac{\partial}{\partial x} \left(\rho \frac{\partial^2 \varphi}{\partial t^2} \right) + \frac{\partial}{\partial y} \left(\rho \frac{\partial^2 \psi_3}{\partial t^2} \right) - \frac{\partial}{\partial z} \left(\rho \frac{\partial^2 \psi_2}{\partial t^2} \right) \\ = (\lambda + \mu) \frac{\partial}{\partial x} \nabla^2 \varphi + \mu \frac{\partial}{\partial x} \nabla^2 \varphi + \mu \frac{\partial}{\partial y} \nabla^2 \psi_3 - \mu \frac{\partial}{\partial z} \nabla^2 \psi_2 \end{aligned}$$

It is easy to see that this equation and the two others from (1-13) written in a similar form will be satisfied if the functions φ and ψ_i are solutions of the equations

$$\nabla^2 \varphi = \frac{1}{\alpha^2} \frac{\partial^2 \varphi}{\partial t^2} \quad \nabla^2 \psi_i = \frac{1}{\beta^2} \frac{\partial^2 \psi_i}{\partial t^2} \quad i = 1, 2, 3 \quad (1-22)$$