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Nonlinear Elasticity

Theory and Applications

Edited by
Y. B. Fu & R. W. Ogden

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Preface

The subject of Finite Elasticity (or Nonlinear Elasticity), although many of its ingredients were available much earlier, really came into its own as a discipline distinct from the classical theory of linear elasticity as a result of the important developments in the theory from the late 1940s associated with Rivlin and the collateral developments in general Continuum Mechanics associated with the Truesdell school during the 1950s and 1960s. Much of the impetus for the theoretical developments in Finite Elasticity came from the rubber industry because of the importance of (natural) rubber in many engineering components, not least car tyres and bridge and engine mountings. This impetus is maintained today with an ever increasing use of rubber (natural and synthetic) and other polymeric materials in a broader and broader range of engineering products. The importance of gaining a sound theoretically-based understanding of the thermomechanical behaviour of rubber was only too graphically illustrated by the role of the rubber O-ring seals in the Challenger shuttle disaster. This extreme example serves to underline the need for detailed characterization of the mechanical properties of different rubberlike materials, and this requires not just appropriate experimental data but also the rigorous theoretical framework for analyzing those data. This involves both elasticity theory *per se* and extensions of the theory to account for inelastic effects.

Over the last few years the applications of the theory have extended beyond the traditional regime of rubber mechanics and they now embrace other materials capable of large elastic strains. These include, in particular, biological tissue such as skin, arterial walls and the heart. This is an important new development and it is increasingly recognized by medical researchers and clinicians that understanding of the mechanics of such tissue is of fundamental importance in developing improved intervention treatments (such as balloon angioplasty) and artificial replacement parts.

While understanding of Finite Elasticity is in itself important the theory also provides a gateway towards the understanding of more complex (non-elastic) material behaviour in the large deformation regime, such as finite deformation plasticity and nonlinear viscoelasticity, and it has an underpinning role in such theories.

Additionally, because of its intrinsic nonlinearity, the equations of Finite Elasticity provide a rich basis for purely mathematical studies in, for example, nonlinear analysis. Indeed, developments, in particular in the theory of partial differential equations, have been stimulated by work in Finite Elasticity, and these *mathematical* developments in turn have had an influence on research in the *mechanical* aspects of Finite Elasticity. Thus, the subject is wide ranging in both its theoretical and application perspectives.

The turn of the century is an appropriate time to assess the state of development of the subject of Finite Elasticity and its potential for further applications and further theoretical development. With this in mind this volume aims to provide an overview of the theory from the perspectives of twelve researchers who have contributed to the subject over the last few years. It is hoped that it will provide a foundation and springboard for possible future developments. The material covered in this volume is necessarily selective and not by any means exhaustive. Thus, some important topics (such as fracture mechanics and the mechanics of composite materials) are not addressed. Nevertheless, there is nothing presently available in the literature that covers such a broad range of topics within the general framework of Finite/Nonlinear Elasticity as that presented here. The various chapters combine concise theory with a number of important applications, and the emphasis is directed more towards understanding of mechanical phenomena and problem solving rather than development of the theory for its own sake.

Different chapters deal with, on the one hand, a number of classical research directions concerned with, for example, exact solutions, universal relations and the effect of internal constraints, and, on the other hand, with recent developments associated with phase transitions and pseudo-elasticity. New ideas from nonlinear analysis, such as nonlinear bifurcation analysis and dynamical systems theory are also featured.

Chapter 1 provides the basic theory required for use in the other chapters. Chapters 2–6 deal with different aspects of the solution of boundary-value problems for unconstrained and internally constrained materials, while Chapters 7 and 8 are concerned with the related topics of membrane theory and the theory of elastic surfaces. Chapter 9 deals with the important topic of non-uniqueness of solution using the tools of singularity theory and bifurcation theory and Chapter 10 examines some related aspects concerned with nonlinear stability analysis based on methods of perturbation theory. Nonlinear dynamics is discussed in Chapter 11, which is concerned with nonlinear wave propagation in an elastic rod. Chapters 12 and 13 are based on different notions of pseudo-elasticity theory: Chapter 12 develops a theory of phase transitions using non-convex strain-energy functions, while Chapter 13 is concerned with the effect of changing the (elastic) constitutive law during the deformation process.

Most of the contributors to this volume participated in an International

Workshop on Nonlinear Elasticity held at City University, Hong Kong, in April 2000. Whilst the chapters in this volume do not form the proceedings of that Workshop it is important to emphasize that this volume and the Workshop were planned in parallel and that the Workshop served to focus ideas for the volume. We are pleased to acknowledge the generous support for the Workshop from the Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong, and the personal support and encouragement of Professor Roderick S. C. Wong, Director of the Centre. We are very grateful to all the contributors for their enthusiastic response to this project and for the timely production of their chapters.

Y.B. Fu
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Elements of the theory of finite elasticity

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In this chapter we provide a brief overview of the main ingredients of the nonlinear theory of elasticity in order to establish the basic background material as a reference source for the other, more specialized, chapters in this volume.

1.1 Introduction

In this introductory chapter we summarize the basic equations of nonlinear elasticity theory as a point of departure and as a reference source for the other articles in this volume which are concerned with more specific topics.

There are several texts and monographs which deal with the subject of nonlinear elasticity in some detail and from different standpoints. The most important of these are, in chronological order of the publication of the first edition, Green and Zerna (1954, 1968, 1992), Green and Adkins (1960, 1970), Truesdell and Noll (1965), Wang and Truesdell (1973), Chadwick (1976, 1999), Marsden and Hughes (1983, 1994), Ogden (1984a, 1997), Ciarlet (1988) and Antman (1995). See also the textbook by Holzapfel (2000), which deals with viscoelasticity and other aspects of nonlinear solid mechanics as well as containing an extensive treatment of nonlinear elasticity. These books may be referred to for more detailed study. Subsequently in this chapter we shall refer to the most recent editions of these works. The review articles by Spencer (1970) and Beatty (1987) are also valuable sources of reference.

Section 1.2 of this chapter is concerned with laying down the basic equations of elastostatics and it includes a summary of the relevant geometry of deformation and strain, an account of stress and stress tensors, the equilibrium equations and boundary conditions and an introduction to the formulation of constitutive laws for elastic materials, with discussion of the important notions of objectivity and material symmetry. Some emphasis is placed on the special case of isotropic elastic materials, and the constitutive laws for anisotropic material consisting of one or two families of fibres are also discussed. The modifications to the constitutive laws when internal constraints such as incompressibility and inextensibility are present are provided. The general boundary-value problem of nonlinear elasticity is then formulated and the circumstances when this can be cast in a variational structure are discussed briefly.

In Section 1.3 some basic examples of boundary-value problems are given. Specifically, the equations governing some homogeneous deformations are highlighted, with the emphasis on incompressible materials. Other chapters in this volume will discuss a range of different boundary-value problems involving non-homogeneous deformations so here we focus attention on just one problem as an exemplar. This is the problem of extension and inflation of a thick-walled circular cylindrical tube. The analysis is given for an incompressible isotropic elastic solid and also for a material with two mechanically equivalent symmetrically disposed families of fibres in order to illustrate some differences between isotropic and anisotropic response.

The (linearized) equations of incremental elasticity associated with small deformations superimposed on a finite deformation are summarized in Section 1.4. The incremental constitutive law for an elastic material is used to identify the (fourth-order) tensor of elastic moduli associated with the stress and deformation variables used in the formulation of the governing equations, and explicit expressions for the components of this tensor are given in the case of an isotropic material. For the two-dimensional specialization, necessary and sufficient conditions on these components for the strong ellipticity inequalities to hold are given for both unconstrained and incompressible materials. A brief discussion of incremental uniqueness and stability is then given in the context of the dead-load boundary-value problem and the associated local inequalities are given explicit form for an isotropic material, again for both unconstrained and incompressible materials. A short discussion of global aspects of non-uniqueness for an isotropic material sets the incremental results in a broader context.

In Section 1.5 the equations of incremental deformations and equilibrium given in Section 1.4 are specialized to the plane strain context in order to provide a formulation for the analysis of incremental plane strain boundary-value problems. Specifically, we provide an example of a typical incremental boundary-value problem by considering bifurcation of a uniformly deformed half-space from a homogeneously deformed configuration into a non-homogeneous local mode of deformation. An explicit bifurcation condition is given for this problem and the results are illustrated for two forms of strain-energy function.

Finally, in Section 1.6 we summarize the equations associated with the (nonlinear) dynamics of an elastic body at finite strain. The (linearized) equations for small motions superimposed on a static finite deformation are then given and these are applied to the analysis of plane waves propagating in a homogeneously deformed material.

References are given throughout the text but these are not intended to provide an exhaustive list of original sources. Where appropriate we mention papers and books where more detailed citations can be found. Also, where a topic is to be dealt with in detail in one of the other chapters of this volume the appropriate citations are included there.

1.2 Elastostatics

In this section we summarize the basic equations of the static theory of nonlinear elasticity, including the kinematics of deformation, the analysis of stress and the governing equations of equilibrium, and we introduce the various forms of constitutive law for an elastic material, including a discussion of isotropy and anisotropy. We then formulate the basic boundary-value problem of nonlinear elasticity. The development here is a synthesis of the essential material taken from the book by Ogden (1997) with some minor differences and additions.

1.2.1 Deformation and strain

We consider a continuous body which occupies a connected open subset of a three-dimensional Euclidean point space, and we refer to such a subset as a *configuration* of the body. We identify an arbitrary configuration as a *reference configuration* and denote this by $\mathcal{B}_r$. Let points in $\mathcal{B}_r$ be labelled by their position vectors $\mathbf{X}$ relative to an arbitrarily chosen origin and let $\partial\mathcal{B}_r$ denote the boundary of $\mathcal{B}_r$. Now suppose that the body is deformed quasi-statically from $\mathcal{B}_r$ so that it occupies a new configuration, $\mathcal{B}$ say, with boundary $\partial\mathcal{B}$. We refer to $\mathcal{B}$ as the *current* or *deformed configuration* of the body. The deformation is represented by the mapping $\chi : \mathcal{B}_r \to \mathcal{B}$ which takes points $\mathbf{X}$ in $\mathcal{B}_r$ to points $\mathbf{x}$ in $\mathcal{B}$. Thus,

$$\mathbf{x} = \chi(\mathbf{X}), \qquad \mathbf{X} \in \mathcal{B}_r, \tag{2.1}$$

where $\mathbf{x}$ is the position vector of the point $\mathbf{X}$ in $\mathcal{B}$. The mapping χ is called the *deformation* from $\mathcal{B}_r$ to $\mathcal{B}$. We require χ to be one-to-one and we write its inverse as χ^{-1}, so that

$$\mathbf{X} = \chi^{-1}(\mathbf{x}), \qquad \mathbf{x} \in \mathcal{B}. \tag{2.2}$$

Both χ and its inverse are assumed to satisfy appropriate regularity conditions. Here, it suffices to take χ to be twice continuously differentiable, but different requirements may be specified in other chapters of this volume.

For simplicity we consider only Cartesian coordinate systems and let $\mathbf{X}$ and $\mathbf{x}$ respectively have coordinates X_α and x_i, where $\alpha, i \in \{1, 2, 3\}$, so that $x_i = \chi_i(X_\alpha)$. Greek and Roman indices refer, respectively, to $\mathcal{B}_r$ and $\mathcal{B}$ and the usual summation convention for repeated indices is used.

The *deformation gradient tensor*, denoted $\mathbf{F}$, is given by

$$\mathbf{F} = \operatorname{Grad}\mathbf{x} \tag{2.3}$$

and has Cartesian components $F_{i\alpha} = \partial x_i/\partial X_\alpha$, Grad being the gradient operator in $\mathcal{B}_r$. Local invertibility of χ requires that $\mathbf{F}$ be non-singular, and we adopt the usual convention that $\det\mathbf{F} > 0$. Similarly, for the inverse deformation gradient

$$\mathbf{F}^{-1} = \operatorname{grad}\mathbf{X}, \quad (\mathbf{F}^{-1})_{\alpha i} = \frac{\partial X_\alpha}{\partial x_i}, \tag{2.4}$$

where grad is the gradient operator in $\mathcal{B}$. With use of the notation defined by

$$J = \det\mathbf{F} \tag{2.5}$$

we then have

$$0 < J < \infty. \tag{2.6}$$

The equation

$$\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X} \tag{2.7}$$

(in components $\mathrm{d}x_i = F_{i\alpha}\mathrm{d}X_\alpha$) describes how an infinitesimal *line element* $\mathrm{d}\mathbf{X}$ of material at the point $\mathbf{X}$ transforms *linearly* under the deformation into the line element $\mathrm{d}\mathbf{x}$ at $\mathbf{x}$.

We now set down how elements of surface area and volume transform. Let $\mathrm{d}\mathbf{A} \equiv \mathbf{N}\mathrm{d}A$ denote a vector surface area element on $\partial\mathcal{B}_r$, where $\mathbf{N}$ is the unit outward normal to the surface, and $\mathrm{d}\mathbf{a} \equiv \mathbf{n}\mathrm{d}a$ the corresponding area element on $\partial\mathcal{B}$. Then, the area elements are connected according to *Nanson's formula*

$$\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A, \tag{2.8}$$

where $\mathbf{F}^{-T} = (\mathbf{F}^{-1})^T$ and ^{T} denotes the transpose. Note that, unlike a line element, the normal vector is not embedded in the material, i.e. $\mathbf{n}$ is not in general aligned with the same line element of material as $\mathbf{N}$.

If $\mathrm{d}V$ and $\mathrm{d}v$ denote volume elements in $\mathcal{B}_r$ and $\mathcal{B}$ respectively then we also have

$$\mathrm{d}v = J\mathrm{d}V. \tag{2.9}$$

For a volume preserving (*isochoric*) deformation we have

$$J = \det \mathbf{F} = 1. \tag{2.10}$$

A material for which (2.10) is constrained to be satisfied for all deformation gradients $\mathbf{F}$ is said to be *incompressible.*

The identities

$$\mathrm{Div}\,(J\mathbf{F}^{-1}) = \mathbf{0}, \quad \mathrm{div}\,(J^{-1}\mathbf{F}) = \mathbf{0} \tag{2.11}$$

are important tools in transformations between equations associated with the reference and current configurations, where Div and div are the divergence operators in $\mathcal{B}_r$ and $\mathcal{B}$ respectively. The first identity in (2.11) can readily be established by integrating (2.8) over an arbitrary closed surface in $\mathcal{B}$ and applying the divergence theorem and the second similarly by integrating $\mathbf{N}\mathrm{d}A$ over an arbitrary closed surface in $\mathcal{B}_r$.

From (2.7) we have

$$|\mathrm{d}\mathbf{x}|^2 = (\mathbf{FM}) \cdot (\mathbf{FM})\,|\mathrm{d}\mathbf{X}|^2 = (\mathbf{F}^T\mathbf{FM}) \cdot \mathbf{M}\,|\mathrm{d}\mathbf{X}|^2, \tag{2.12}$$

where we have introduced the unit vector $\mathbf{M}$ in the direction of $\mathrm{d}\mathbf{X}$ and $\cdot$ signifies the scalar product of two vectors. Then, the ratio $|\mathrm{d}\mathbf{x}|/|\mathrm{d}\mathbf{X}|$ of the lengths of a line element in the deformed and reference configurations is given by

$$\frac{|\mathrm{d}\mathbf{x}|}{|\mathrm{d}\mathbf{X}|} = |\mathbf{FM}| = [\mathbf{M} \cdot (\mathbf{F}^T\mathbf{FM})]^{1/2} \equiv \lambda(\mathbf{M}). \tag{2.13}$$

Equation (2.13) defines the *stretch* $\lambda(\mathbf{M})$ *in the direction* $\mathbf{M}$ at $\mathbf{X}$, and we note that it is restricted according to the inequalities

$$0 < \lambda(\mathbf{M}) < \infty. \tag{2.14}$$

If there is no stretch in the direction $\mathbf{M}$ then $\lambda(\mathbf{M}) = 1$ and hence

$$(\mathbf{F}^T\mathbf{FM}) \cdot \mathbf{M} = 1. \tag{2.15}$$

If there is no stretch in any direction, i.e. (2.15) holds for all $\mathbf{M}$, then the material is said to be *unstrained* at $\mathbf{X}$, and it follows that $\mathbf{F}^T\mathbf{F} = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor. A suitable tensor measure of strain is therefore $\mathbf{F}^T\mathbf{F} - \mathbf{I}$ since this tensor vanishes when the material is unstrained. This leads to the definition of the *Green strain tensor*

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}), \tag{2.16}$$

where the 1/2 is a normalization factor. If, for a given $\mathbf{M}$, equation (2.15) holds for all possible deformation gradients $\mathbf{F}$ then the considered material is said to be *inextensible* in the direction $\mathbf{M}$.

The deformation gradient can be decomposed according to the *polar decompositions*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{2.17}$$

where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{U}$, $\mathbf{V}$ are positive definite and symmetric tensors. Each of the decompositions in (2.17) is unique. Respectively, $\mathbf{U}$ and $\mathbf{V}$ are called the *right* and *left stretch tensors*.

These stretch tensors can also be put in spectral form. For $\mathbf{U}$ we have the *spectral decomposition*

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{2.18}$$

where $\lambda_i > 0$, $i \in \{1,2,3\}$, are the *principal stretches*, $\mathbf{u}^{(i)}$, the (unit) eigenvectors of $\mathbf{U}$, are called the *Lagrangian principal axes* and $\otimes$ denotes the tensor product. Note that $\lambda(\mathbf{u}^{(i)}) = \lambda_i$ in accordance with the definition (2.13). Similarly, $\mathbf{V}$ has the spectral decomposition

$$\mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.19}$$

where

$$\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}, \quad i \in \{1,2,3\}. \tag{2.20}$$

It follows from (2.5), (2.17) and (2.18) that

$$J = \lambda_1 \lambda_2 \lambda_3. \tag{2.21}$$

Using the polar decompositions (2.17) for the deformation gradient $\mathbf{F}$, we may also form the following tensor measures of deformation:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2. \tag{2.22}$$

These define $\mathbf{C}$ and $\mathbf{B}$, which are called, respectively, the *right* and *left Cauchy-Green deformation tensors*.

More general classes of strain tensors, i.e. tensors which vanish when there is no strain, can be constructed on the basis that $\mathbf{U} = \mathbf{I}$ when the material is unstrained. Thus, for example, we define Lagrangian strain tensors

$$\mathbf{E}^{(m)} = \frac{1}{m}(\mathbf{U}^m - \mathbf{I}), \quad m \neq 0, \tag{2.23}$$

$$\mathbf{E}^{(0)} = \ln \mathbf{U}, \quad m = 0, \tag{2.24}$$

where m is a real number (not necessarily an integer). Eulerian strain tensors based on the use of $\mathbf{V}$ may be constructed similarly. See, for example, Doyle and Ericksen (1956), Seth (1964) and Hill (1968, 1970, 1978). Note that for $m = 2$ equation (2.23) reduces to the Green strain tensor (2.16). For discussion of the logarithmic strain tensor (2.24) we refer to, for example, Hoger (1987).

Let ρ_r and ρ be the *mass densities* in $\mathcal{B}_r$ and $\mathcal{B}$ respectively. Then, since the material in the volume element $\mathrm{d}V$ is the same as that in $\mathrm{d}v$ the mass is conserved, i.e. $\rho\mathrm{d}v = \rho_r\mathrm{d}V$, and hence, from (2.9), we may express the *mass conservation equation* in the form

$$\rho_r = \rho J. \tag{2.25}$$

1.2.2 Stress tensors and equilibrium equations

The surface force per unit area (or *stress vector*) on the vector area element $\mathrm{d}\mathbf{a}$ is denoted by $\mathbf{t}$. It depends on $\mathbf{n}$ according to the formula

$$\mathbf{t} = \boldsymbol{\sigma}^T\mathbf{n}, \tag{2.26}$$

where $\boldsymbol{\sigma}$, a second-order tensor independent of $\mathbf{n}$, is called the *Cauchy stress tensor.*

By means of (2.8) the force on $\mathrm{d}\mathbf{a}$ may be written as

$$\mathbf{t}\mathrm{d}a = \mathbf{S}^T\mathbf{N}\mathrm{d}A, \tag{2.27}$$

where the *nominal stress tensor* $\mathbf{S}$ is related to $\boldsymbol{\sigma}$ by

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}. \tag{2.28}$$

The *first Piola-Kirchhoff stress tensor*, denoted here by $\boldsymbol{\pi}$, is given by $\boldsymbol{\pi} = \mathbf{S}^T$ and this will be used in preference to $\mathbf{S}$ in some parts of this volume.

Let $\mathbf{b}$ denote the body force per unit mass. Then, in integral form, the *equilibrium equation* for the body may be written with reference either to $\mathcal{B}$ or $\mathcal{B}_r$. Thus,

$$\int_{\mathcal{B}} \rho\mathbf{b}\mathrm{d}v + \int_{\partial\mathcal{B}} \boldsymbol{\sigma}^T\mathbf{n}\mathrm{d}a = \int_{\mathcal{B}_r} \rho_r\mathbf{b}\mathrm{d}V + \int_{\partial\mathcal{B}_r} \mathbf{S}^T\mathbf{N}\mathrm{d}A = \mathbf{0}. \tag{2.29}$$

On use of the divergence theorem equations (2.29) yield the equivalent equilibrium equations

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \mathbf{0}, \tag{2.30}$$

$$\operatorname{Div}\mathbf{S} + \rho_r\mathbf{b} = \mathbf{0}, \tag{2.31}$$

where again div and Div denote the divergence operators in $\mathcal{B}$ and $\mathcal{B}_r$ respectively. The derivation of the pointwise equations (2.30) and (2.31) requires