TENSORS IN ELECTRICAL ENGINEERING

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Tensors in Electrical Engineering

by

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Preface

The unified theory of electrical machines was developed in 1934 by Gabriel Kron. His fundamental step in building this theory was to consider machines as systems of inductively coupled coils and to relate each machine to a basic primitive form. This technique involved first, the idea of a group or family of machines, second the transformation of quantities in the primitive machine and third, the invariance of power under transformation. The branch of mathematics required for such a procedure was already a century old at that time, namely tensor analysis applied to geometry (and later to dynamics by Synge). Kron developed his own technique in applying tensors to electrical machines and networks. In recent years attempts have been made to simplify the generalised theory by removing the concepts of tensors, the machine equations being handled by matrix algebra. The author is of the opinion that the tensor concepts should not be lost in a physically obscure matrix process, since it is the tensors which apply to all machines in all reference frames. The situation has been described by Lewis Carroll, in circumstances perhaps not inappropriate; "It's the same thing you know" said Alice. "It's not the same thing a bit" replied the Hatter.

The present book is written to give post-graduate students, lecturers and research workers in electrical machine dynamics a survey of Kron's application of tensors and to describe the methods by which circuits, fields, and dynamo action are being united in one mathematical discipline.

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J. W. L.

List of Symbols

The book deals with applications of tensor analysis to several branches of engineering. It has been found that the equations in each field of application are more easily distinguished when they are written in the generally accepted notation and therefore complete standardisation of the symbols has not been attempted. There is, in fact, no confusion as the applications of the symbols in each section are self-evident. The following list gives the groups of symbols used.

(i) *Geometrical*

u^α = general curvilinear co-ordinates
$g_{\alpha\beta}$ = system metric tensor
$\mathbf{i}_\alpha$ = unit vectors
$\mathbf{a}_\alpha$ = unitary vectors
s = displacement of a point
F_α = physical components of a vector
f_α = covariant components of a vector
σ = element of surface area

(ii) *Machines*

(*a*) indices

a, b, c = quantities in axes fixed to machine stator and rotor windings
k, n, m = quantities in axes all relatively stationary
α, β, γ = quantities in axes fixed or free on the stator, rotating freely on the rotor
s, t = quantities associated with mechanical rotational effects (e.g. generated voltage and torque)
u, v, w = quantities in the general equation

(*b*) symbols

f_m, f_α, etc. = electrical voltage vectors
x^k, x^α, etc. = electrical variables, the electrical charges in machine windings
$\dot{x}^\alpha (= i^\alpha)$ = electrical current
$R_{\gamma\alpha}$ = resistance matrix
$L_{\gamma\alpha}$ = inductance matrix
$G_{\gamma\alpha}$ = matrix of generated voltage coefficients
x^s = mechanical variable θ

$\dot{x}^s$ = angular velocity of machine rotor
$L_{ss}(= J)$ = moment of inertia of rotor
C_α^k = transformation matrix
$\Omega_{\alpha\beta}^\delta$ = 'non-holonomic object' containing functions of C
$[\alpha\beta, \gamma]$ = Christoffel symbol, a connection term containing functions of $L_{\alpha\beta}$
$\Gamma_{\alpha\beta.\gamma}$ = connection term containing functions of $L_{\alpha\beta}$ and $\Omega_{\alpha\beta.\gamma}$
B_α = flux density matrix

(iii) *Electromagnetic quantities*

E_α = electric field vector
H_α = magnetic field vector
B_α = flux density
I^α = open-mesh (junction-pair) current
i^α = closed-mesh current
J^α = current density
D^α = electric displacement
ρ = electric charge density
$\sigma_{\alpha\beta}$ = conductivity components
$\mu^{\alpha\beta}$ = permeability components
$\varepsilon^{\alpha\beta}$ = permittivity components

(iv) *Fluid flow*

$\boldsymbol{\psi}$ = vector potential (stream function, where curl $\boldsymbol{\psi} = \rho\mathbf{v}$)
$\mathbf{v}$ = velocity vector
$\boldsymbol{\Gamma}$ = vorticity (= curl ρ^{-1} curl $\boldsymbol{\psi}$)
ρ = density

(v) *Elasticity*

$\sigma^{\alpha\beta}$ = stress components
$e_{\alpha\beta}$ = strain components
s_α = displacement components
$\left.\begin{matrix}\lambda\\ \mu\end{matrix}\right\}$ = Lamé's Constants
f^α = force vector

(vi) *Neutron diffusion*

$\mathbf{D}$ = diffusion coefficient
n = neutron density
S = neutron source strength
ϕ = neutron flux = nv
$\mathbf{v}$ = neutron velocity

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APPENDICES

CHAPTER I

Determinants and Matrices

1.1 Determinants

The algebraic symbol x implies a range of arithmetical values. The manipulation of symbols in algebraic processes shows how, ultimately, any of the implied arithmetical figures will be correlated in a particular case. In applied mathematics one usually employs symbols throughout an analysis and then, when the final relationship is established, one investigates the behaviour of the system for a range of numbers. Matrix algebra[1] is an extension of this to deal systematically with groups of several sets of algebraic symbols when the sets are governed by the same laws simultaneously, the individual symbols having different implied numerical values. For example, consider the two simultaneous linear equations

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 \\ y_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{1.1}$$

The only difference between the first and second of these equations is that they will have different arithmetical values. In matrix algebra the sets of symbols are grouped into a more compact notation. The equations (1.1) are written, in matrix notation,

$$y_m = \sum_{n=1}^{n=2} a_{mn}x_n \qquad (m, n = 1, 2) \tag{1.2}$$

It is found that when an index is repeated (e.g. index n), then summation is almost always implied, and it is usual to dispense with the corresponding summation sign and write

$$y_m = a_{mn}x_n \tag{1.3}$$

Various operations of algebra and calculus can be carried out on matrices such as those shown in equation (1.3). The end result can be expanded to give the final set of equations, and these can then be investigated numerically. The numerical solution of simultaneous equations is often carried out by the processes of the theory of determinants,[1] of which a brief summary of the elementary rules is as follows.

Equation in one unknown,

$$y = ax \tag{1.4}$$

Solution:

$$x = y/a \tag{1.5}$$

Equations in two unknowns,

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 \\ y_2 &= a_{21}x_1 + a_{22}x_2 \end{aligned} \tag{1.5}$$

Solution:

$$x_1 = \frac{y_1 a_{22} - y_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \qquad x_2 = \frac{y_2 a_{11} - y_1 a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \tag{1.7}$$

The combination of products on the R.H.S. in each case can be selected by inspection of the coefficients in the equations when the selection pattern is known. The above solution can be written

$$x_1 = \frac{\begin{vmatrix} y_1 & a_{12} \\ y_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & y_1 \\ a_{21} & y_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \tag{1.8}$$

The sets of values enclosed by vertical lines are called determinants. It will be seen that the denominator determinant is made up of the coefficients while in the numerator the variables y replace the column corresponding to the appropriate variable x. This is known as Cramer's Rule. The diagonal quantities are multiplied and summed with the signs shown in equations (1.7).

Equations in three unknowns,

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \tag{1.9}$$

Solution:

$$x_1 = \frac{\begin{vmatrix} y_1 & a_{12} & a_{13} \\ y_2 & a_{22} & a_{23} \\ y_3 & a_{32} & a_{33} \end{vmatrix}}{\Delta} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & y_1 & a_{13} \\ a_{21} & y_2 & a_{23} \\ a_{31} & y_3 & a_{33} \end{vmatrix}}{\Delta} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & y_1 \\ a_{21} & a_{22} & y_2 \\ a_{31} & a_{32} & y_3 \end{vmatrix}}{\Delta}$$

where

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \tag{1.10}$$

The third order determinant is expanded as follows. The first two columns are repeated and diagonal products are formed, for example,

$$\begin{array}{ccccc} y_1 & a_{12} & a_{13} & y_1 & a_{12} \\ y_2 & a_{22} & a_{23} & y_2 & a_{23} \\ y_3 & a_{32} & a_{33} & y_3 & a_{32} \end{array} \tag{1.11}$$

giving

$$y_1a_{22}a_{33} + a_{12}a_{23}y_3 + a_{13}y_2a_{32} \tag{1.12}$$

The process is repeated, starting from y_3,

$$y_3a_{22}a_{13} + a_{32}a_{23}y_1 + a_{33}y_2a_{12} \tag{1.13}$$

The second set is subtracted from the first,

$$\begin{vmatrix} y_1 & a_{12} & a_{13} \\ y_2 & a_{22} & a_{23} \\ y_3 & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} y_1a_{22}a_{23} + a_{12}a_{23}y_3 + a_{13}y_2a_{32} \\ -y_3a_{22}a_{13} - a_{32}a_{23}y_1 - a_{33}y_2a_{12} \end{array} \tag{1.14}$$

Equations in four unknowns,

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ y_2 &= a_{12}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ y_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \end{aligned} \tag{1.15}$$

Solution:

$$x_1 = \frac{\begin{vmatrix} y_1 & a_{12} & a_{13} & a_{14} \\ y_2 & a_{22} & a_{23} & a_{24} \\ y_3 & a_{32} & a_{33} & a_{34} \\ y_4 & a_{42} & a_{43} & a_{44} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}} \quad \text{etc.} \tag{1.16}$$

The fourth order determinant is expanded as the sum of four products. Each of these is the product of the element of a row and the third order determinant remaining when that row and the corresponding column are eliminated. Each of the smaller determinants, called 'Minors', has its sign determined by $(-1)^{i+j}$ where i and j are the numbers of the row and column that have been eliminated. The signed minor is called the 'co-factor'. The determinant can be expanded as the sum of the products of the elements of any row or column and the corresponding co-factors.

The general expression for expansion of an n^{th} order determinant can be written

$$\Delta_n = a_{i1}M_{i1}(-1)^{i+1} + a_{i2}M_{i2}(-1)^{i+2} + \text{etc.} \tag{1.17}$$

or

$$\Delta_n = a_{(i)j}M_{(i)j}(-1)^{i+j} \tag{1.18}$$

The expansion of the fourth order determinant in the numerator of equation (1.16) is therefore

$$y_1\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - y_2\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + y_3\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - y_4\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \tag{1.19}$$

Every numerical determinant when expanded has a single arithmetical value, which may be positive, negative or zero. If a determinant has the value zero then the rows and columns are not completely independent. Some further properties of determinants used in the solution of simultaneous equations are as follows—

(*a*) The value of a determinant is unaltered by transposition of rows and columns.

(*b*) If two rows or columns are interchanged the sign of the determinant is reversed.

(*c*) If two rows or columns are equal or proportional, term by term, the value of the determinant is zero.

(*d*) If the rows and columns are dependent, the value of the determinant is zero.

(*e*) If the elements of any row or column are multiplied by a constant, the determinant is multiplied by that constant.

(*f*) If each element of a row or column is the sum of two terms the determinant can be expressed as the sum of two determinants.

(*g*) If the elements of any row or column are increased or decreased by equimultiples of the corresponding elements of any other row or column the value of the determinant is unchanged.

(*h*) The value of a triangular determinant or a diagonal determinant is the product of the diagonal elements.

1.2 Matrices

In large complicated calculations coefficients and variables may often be grouped in separate blocks, as is the case in setting up determinants. However, a matrix is only an array of numbers, and the matrix does not have a single numerical value as the determinant does. The matrix grouping of terms is used merely to simplify routine operations. Matrices may be added together, multiplied, differentiated, etc., the operations being carried out on each of the individual terms, but some of these operations can be carried out only when the several matrices concerned have the proper compatible numbers of rows and columns.

The set of simultaneous equations

$$\begin{aligned} e_1 &= Z_{11}i_1 + Z_{12}i_2 + Z_{13}i_3 \\ e_2 &= Z_{21}i_1 + Z_{22}i_2 + Z_{23}i_3 \\ e_3 &= Z_{31}i_1 + Z_{32}i_2 + Z_{33}i_3 \end{aligned} \tag{1.20}$$

can be written in matrix form

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \cdot \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} \tag{1.21}$$

In index notation,

$$e_m = Z_{mn}i_n \tag{1.22}$$

Another way of writing equations (1.21) is

$$\underset{m\searrow}{\boxed{\begin{array}{c} e_1 \\ \hline e_2 \\ \hline e_3 \end{array}}} = \underset{m\searrow n}{\boxed{\begin{array}{c|c|c} Z_{11} & Z_{12} & Z_{13} \\ \hline Z_{21} & Z_{22} & Z_{23} \\ \hline Z_{31} & Z_{32} & Z_{33} \end{array}}} \cdot \underset{n\searrow}{\boxed{\begin{array}{c} i_1 \\ \hline i_2 \\ \hline i_3 \end{array}}} \tag{1.23}$$

The repeated index n is summed. Thus the two matrices on the right hand side are multiplied by starting at the top left-hand element Z_{11}, multiplying it by i_1, proceeding to $Z_{12}i_2$, etc., and summing these products. This gives the first row. Similar operations give the remaining rows.

Addition and subtraction

Obviously two matrices can have their elements added or subtracted only if they have the same number of rows and columns.

Multiplication by a constant

A matrix is multiplied by a constant when each of its terms is multiplied by that constant.

Multiplication of matrices

Two matrices may be multiplied together only if one of them has the same number of rows as the number of columns in the other. The order of multiplication is important. This is most clearly shown by examples.

Matrix **B** post-multiplied by matrix **A** is written **B** . **A**. This is not the same as **A** . **B** (which would read **B** pre-multiplied by **A**) and it may not in fact be possible to carry out the second multiplication, for example,

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} Ax + By + Cz \\ Dx + Ey + Fz \end{bmatrix} \qquad (1.24)$$

Obviously in this case the multiplication could not be carried out if the order of multiplication were reversed.

Again,

$$\begin{bmatrix} A & B & C \\ D & E & F \end{bmatrix} \begin{bmatrix} x & s \\ y & t \\ z & u \end{bmatrix} = \begin{bmatrix} Ax + By + Cz & As + Bt + Cu \\ Dx + Ey + Fz & Ds + Et + Fu \end{bmatrix} \qquad (1.24a)$$

This could be described as 'two by three' by 'three by two'. The adjacent numbers show the compatibility of the matrices and the outer numbers give the number of rows and columns of the product matrix. With square matrices the order of multiplication must be clearly specified.

Occasionally it is necessary to operate with matrices that have the rows and columns transposed. It is seen that the transpose of (**AB**) = (transpose of **B**) (transpose of **A**); for example,

$$\underset{(\mathbf{A})}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}} \cdot \underset{(\mathbf{B})}{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 2 \\ 5 & 6 & 3 \end{bmatrix}} = \underset{(\mathbf{AB})}{\begin{bmatrix} 22 & 28 & 14 \\ 49 & 64 & 32 \end{bmatrix}} \tag{1.25}$$

$$\underset{(\mathbf{B}_t)}{\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}} \cdot \underset{(\mathbf{A}_t)}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}} = \underset{(\mathbf{B}_t\mathbf{A}_t)}{\begin{bmatrix} 22 & 49 \\ 28 & 64 \\ 14 & 32 \end{bmatrix}} \tag{1.26}$$

Any matrix can be written as the sum of a symmetrical matrix and a skew-symmetrical matrix. In index notation,

$$A_{ij} = \underset{\text{Symm.}}{B_{ij}} + \underset{\text{Skew-symm.}}{C_{ij}} \tag{1.27}$$

where
$$B_{ij} = \tfrac{1}{2}(A_{ji} + A_{ij}) \tag{1.28}$$

and
$$C_{ij} = \tfrac{1}{2}(A_{ji} - A_{ij}) \tag{1.29}$$

For example,

$$A_{ij} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad B_{ij} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} \quad C_{ij} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Matrix inversion

Division by a matrix can be carried out only by use of the inverse matrix. The solution of the set of equations (1.20) is written

$$i_n = (Z_{mn})^{-1} e_m \tag{1.30}$$

and the inverse of the matrix of coefficients in (1.23) is required. A matrix has an inverse only if it is square ('non-singular') and this is found as follows.

(1) Transpose the matrix.

(2) Replace each element by its co-factor (the resulting matrix is called the 'adjoint').

(3) Divide each element of this matrix by the determinant of the original matrix.

$$\mathbf{A}^{-1} = \frac{adj\mathbf{A}}{|\mathbf{A}|} \tag{1.31}$$

It is evident that the process is tedious by hand calculation if the matrix has more than three rows and columns. However, a large matrix can be partitioned and systematically solved in stages.[3,4] Special techniques have also been developed for inverting matrices and electronic computers have standard programmes for this purpose.

Partitioning

Any matrix can be sub-divided into several sections, each section forming an element of the partitioned matrix. Care is required in handling partitioned matrices since in some cases operations carried out on the whole matrix must also be carried out on the smaller matrices comprising the elements, for example when transposing,

$$\mathbf{A}' = \left[\begin{array}{cc|cc} 1 & 2 & 5 & 6 \\ 3 & 4 & 7 & 8 \\ \hline 9 & 10 & 13 & 14 \\ 11 & 12 & 15 & 16 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{1.32}$$

$$\mathbf{A}_t' = \left[\begin{array}{c|c} A_t & C_t \\ \hline B_t & D_t \end{array}\right] \tag{1.33}$$

If two partitioned matrices are to be multiplied, the partitioned parts must be compatible as to rows and columns; for example,

$$\left[\begin{array}{cc|c} a & b & c \\ d & e & f \\ \hline g & h & i \end{array}\right] \cdot \left[\begin{array}{c} x \\ y \\ \hline z \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cc|c} a & b & c \\ \hline d & e & f \\ g & h & i \end{array}\right] \cdot \left[\begin{array}{c} x \\ y \\ \hline z \end{array}\right]$$

or

$$\left[\begin{array}{c|cc} a & b & c \\ d & e & F \\ \hline g & h & j \end{array}\right] \cdot \left[\begin{array}{c} x \\ \hline y \\ z \end{array}\right]$$

In each case

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \cdot \left[\begin{array}{c} p \\ \hline q \end{array}\right] = \left[\begin{array}{c} Ap + Bq \\ \hline Cp + Dq \end{array}\right] \qquad (1.34)$$

Elimination of rows and columns[4]

A matrix equation can be partitioned across the variables to be eliminated. This forms compound matrices

$$\left[\begin{array}{c} \mathbf{e}_1 \\ \hline \mathbf{e}_2 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{Z}_1 & \mathbf{Z}_2 \\ \hline \mathbf{Z}_3 & \mathbf{Z}_4 \end{array}\right] \cdot \left[\begin{array}{c} \mathbf{i}_1 \\ \hline \mathbf{i}_2 \end{array}\right] \qquad (1.35)$$

and since the whole **Z**-matrix is square the matrices $\mathbf{Z}_1$ and $\mathbf{Z}_4$ will also be square. The elements of the matrix equation (1.35) are all matrices.

Suppose it is required to eliminate the variables represented in (1.35) by $\mathbf{i}_2$.

Expanding the equation (1.35)

$$\mathbf{e}_1 = \mathbf{Z}_1\mathbf{i}_1 + \mathbf{Z}_2\mathbf{i}_2$$

$$\mathbf{e}_2 = \mathbf{Z}_3\mathbf{i}_1 + \mathbf{Z}_4\mathbf{i}_2 \qquad (1.36)$$

and, eliminating $\mathbf{i}_2$

$$\mathbf{Z}_4\mathbf{i}_2 = \mathbf{e}_2 - \mathbf{Z}_3\mathbf{i}_1$$

$$\mathbf{i}_2 = \mathbf{Z}_4^{-1}(\mathbf{e}_2 - \mathbf{Z}_3\mathbf{i}_1) \qquad (1.37)$$

$$\mathbf{e}_1 = \mathbf{Z}_1\mathbf{i}_1 + \mathbf{Z}_2\mathbf{Z}_4^{-1}(\mathbf{e}_2 - \mathbf{Z}_3\mathbf{i}_1)$$

$$= \mathbf{Z}_2\mathbf{Z}_4^{-1}\mathbf{e}_2 + (\mathbf{Z}_1 - \mathbf{Z}_2\mathbf{Z}_4^{-1}\mathbf{Z}_3)\mathbf{i}_1 \qquad (1.38)$$