TAK1NG SUDOKU SER1OUSLY

The Math Behind the World's Most Popular Pencil Puzzle

JASON ROSENHOUSE AND LAURA TAALMAN

" Many people around the globe enjoy Sudoku. Laura Taalman and Jason Rosenhouse understand the entertaining aspects of the Sudoku craze and, in this refreshing and delightful book, have analyzed the puzzle from the perspective of mathematics for the first time."

Maki Kaji, Godfather of Sudoku

" Rosenhouse and Taalman have magically transformed the 9x9 Sudoku grid into a window through which we can see how mathematicians think and how mathematics really works. It's hard to imagine that any other book could capture the heart of mathematics so thoroughly, accurately, and vividly. After all, solving puzzles, via a mix of established techniques and MacGyver-esque improvisation, is what both mathematics and Sudoku are all about."

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" In a clear and entertaining style, the authors show how mathematics can improve your understanding of Sudoku. But more importantly, they demonstrate how Sudoku can also improve your understanding of mathematics."

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Taking Sudoku Seriously

In memory of Martin Gardner, who showed a generation of mathematicians the value of puzzles as a gateway into mathematics.

PREFACE

Every math teacher knows the frustration of directing a seemingly simple question to a class and receiving blank stares in return. In part, this reaction can be attributed to general student apathy or to a fear of giving the wrong answer. There is, however, a more fundamental issue to be addressed.

Most people, when asked to describe mathematics, will talk about the tedious algorithms of arithmetic or the seemingly arbitrary rules of algebra. For them it is all about symbol manipulation and mindless computation. This view is entirely understandable given that they probably saw little else in their grade school and high school mathematics classes.

Mathematicians do not recognize their discipline in such descriptions. We see arithmetic and algebra as tools used in doing mathematics, just as hammers and handsaws are tools used in carpentry. For professionals, mathematics is about curiosity, imagination, and solving problems. There are questions that are instinctive and natural for mathematicians that rarely occur to those looking in from the outside. There is such a thing as a mathematical view of the world. Sadly, it is a view that is too often hidden from those struggling to learn the subject.

Which brings us back to the blank stares. Often the problem is simply that mathematicians have a way of expressing themselves that makes little sense to those outside the club. Students unaccustomed to the sorts of questions mathematicians ask, or unaware that mathematics is about asking questions in the first place, will often be confused by questions more experienced people regard as simple. We need first to develop mathematical thinking in our students before we expect them to toss off answers to our questions.

That is where the Sudoku puzzles come in. We can define a *Sudoku square* as a 9×9 grid in which every row, column, and 3×3 block contains the digits 1–9 exactly once. A *Sudoku puzzle* is then a square in which some of the cells have been filled in while others are blank. The goal of the solver is to fill in the blank cells in such a way that the result is a Sudoku square. If the puzzle is sound there will only be one way of doing that.

Here is an example. This is a level 3 puzzle, where level 1 is the easiest and level 5 is the hardest.

Puzzle 1: Sudoku Warm-Up.

Fill in the grid so that each row, column, and block contains each of the numbers 1–9 exactly once. The solution to this puzzle is at the end of the book.

	2	3						
	5		7				8	3
		4				2		7
				7			6	
			9		6			
	3			2				
3		5				8		
6	1				8		7	
						4	1	

Over the past five years, Sudoku puzzles have become a mainstay of many newspapers. Such venues are typically careful to assure the reader that, the presence of numbers notwithstanding, Sudoku puzzles are not math problems. They are keen to stress that any collection of nine distinct symbols, such as the first nine letters of the alphabet, would work just as well.

This sort of thing sounds bizarre to a mathematician. In saying that Sudoku does not involve mathematics, the newspaper really means it does not involve arithmetic. The sort of reasoning that goes in to solving a Sudoku puzzle, on the other hand, is at the heart of what mathematics is all about. That so many people will claim to hate doing mathematics while simultaneously enjoying the challenge of solving a puzzle is a source of frustration to those of us in the business.

To a mathematician, Sudoku puzzles immediately suggest a whole host of interesting questions even beyond the reasoning that goes in to solving them. How many Sudoku squares are there? What sorts of transformations can you do to a Sudoku square to produce other such squares? What is the smallest number of initial clues a sound puzzle can have? What is the largest number of initial clues a puzzle can have without having a unique solution? Is it possible to have a Sudoku square in which each 3×3 block is actually a semimagic square (so that the digits in each row and column within the block add up to the same sum)? In attempting to answer these questions we will inevitably encounter a lot of interesting mathematics.

More than that, however, we will use Sudoku puzzles and their variants as a gateway into mathematical thinking generally. This is both a math book and a puzzle book. The puzzles, in addition to being enjoyable simply as stand-alone brainteasers, will serve to complement and introduce the mathematical concepts in the text. Our emphasis throughout is on asking questions and solving problems; technical mathematical machinery will be introduced only as it arises naturally in the course of our reasoning.

We have a number of different audiences in mind. For students in high school or college we intend to provide a view of mathematics that is very different from what is usually presented. It is a far more realistic view than the one implied by years of training in tedious symbol manipulation. For educators we hope to provide some novel ideas for how to bring genuine mathematical thinking into the classroom in a context that will be interesting and accessible to students. For any layperson with a general interest in mathematics, we provide plenty of food for thought and intellectual stimulation. Professional mathematicians can benefit from seeing familiar mathematical abstractions applied in novel settings.

We have assumed little beyond high school mathematics. Indeed, if you flip through the book right now you will notice that for the most part we make limited use of mathematical symbols. Our focus is on ideas and reasoning; "notions, not notations," as the saying goes. That is not to say, however, that the book is always easy going. Mathematics takes some getting used to, and you should not be surprised if you have to pause periodically to mull over something we have said. Furthermore, things do get gradually more complex as we go along, and readers without previous mathematical experience might find some of the concluding material a bit more challenging than what came before. Even here, though, we believe we have provided enough commentary to make the central ideas comprehensible to all. In those few cases where we have elected to include some more technical material, the dense calculations can be skimmed over without losing the thread of the discussion.

The book is structured as follows: In the first chapter we examine techniques for solving Sudoku puzzles and discuss the general question of what constitutes a math problem. Chapter 2 discusses the notion of a Latin square, an object of long-standing interest to mathematicians of which Sudoku squares are a special case. Chapter 3 discusses Greco-Latin squares, which are an extension of the idea of a Latin square. Chapters 4 and 5 discuss two counting problems related to Sudoku. Specifically, we determine the total number of Sudoku squares and the total number of "fundamentally different" squares. In the course of this discussion, we cannot avoid presenting fundamental ideas from combinatorics and abstract algebra. Chapter 6 presents the problem of how one finds interesting Sudoku puzzles and places this problem within the context of search problems generally. Chapters 7 and 8 investigate connections between Sudoku, graph theory, and polynomials. Chapter 9 is an exploration of Sudoku extremes. We look for puzzles with the maximal number of vacant regions, with the minimal number of starting clues, and numerous others. The book concludes with a gallery of novel Sudoku variations. No math here, just pure solving fun! All of the puzzles presented in the text, save for a handful of exceptions that are explicitly identified, are original to this volume.

A final, bureaucratic detail. The solutions to many of the puzzles appear in the back of the book. In some cases, however, the solution to the puzzle is essential to the exposition and, therefore, has been included in the text. Wherever possible, we have placed the solution to a puzzle on a different page from the puzzle itself. Occasionally this was not possible. For that reason you may find it useful to read with an index card in hand. This will allow you to conceal portions of the page you do not wish to read immediately.

The history of math and science shows there is often great insight to be gained from the earnest consideration of trivial pursuits. Probability theory is today an indispensable tool in many branches of science, but it was born out of gambling and games of chance. In the early days of computer science and artificial intelligence, much attention was given to the relatively unimportant problem of programming a computer to play chess.

We have similar ambitions for this book. Perhaps you have tended to see Sudoku puzzles as an amusing diversion, useful only for passing the time during long airplane rides. After reading this book you will see instead a gateway into the world of mathematics. It is a far different, and more beautiful, world than you may think.

The authors would like to thank Philip Riley, whose computer prowess assisted greatly with the construction of many of the Sudoku puzzles in this book. Without Phil's work at Brainfreeze Puzzles, large portions of this book would not exist. We would also like to thank our Sudoku Master beta-tester Rebecca Field for checking all of the puzzles in the text for accuracy and playability. Finally, we would like to thank Phyllis Cohen, our editor at Oxford University Press, who was tremendously helpful and supportive throughout the writing of this book.

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