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Domain Decomposition Methods for Partial Differential Equations

ALFIO QUARTERONI
and ALBERTO VALLI

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ALFIO QUARTERONI

Professor of Numerical Analysis, Politecnico di Milano, Italy, and École Polytechnique Fédérale de Lausanne, Switzerland

ALBERTO VALLI

Professor of Functional Analysis, University of Trento, Italy

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PREFACE

One must never exhaust a subject to the point
that the reader has nothing left to do.
It is not a matter of inducing to read, but of inducing to think.

Charles-Louis de Montesquieu

The numerical solution of differential problems of practical interest may often lead to large-scale algebraic systems. Modern supercomputers make it possible nowadays to afford a wide range of problems that were unaffordable until recently. However, the size of most of such problems is so large that substantial attention needs to be paid to the improvement of existing numerical algorithms, as well as to the development of new ones that may better fit the architecture of available supercomputers.

Domain decomposition for the numerical solution of partial differential equations is a relatively new field (the first important ideas emerged in the early eighties). In particular, it is one of the most significant ways for devising parallel algorithms that can benefit strongly from multiprocessor computers. Parallel approaches are mandatory for very large-scale numerical problems like those that arise very often in many branches of physics and engineering.

Any domain decomposition method is based on the assumption that the given computational domain, say Ω, is partitioned into subdomains Ω_i, $i = 1, \ldots, M$, which may or may not overlap. Next, the original problem can be reformulated upon each subdomain Ω_i, yielding a family of subproblems of reduced size that are coupled one to another through the values of the unknown solution at subdomain interfaces.

Very often the interface coupling is removed at the expense of introducing an iterative process among subdomains, yielding at each step independent subproblems (of lower complexity) upon subdomains, which can be efficiently faced by multiprocessor systems.

When properly devised, these iterative procedures intrinsically embody a preconditioner for the system induced on the interface unknowns. A distinguishing feature of a domain decomposition method is the property of optimality of such a preconditioner; that is, its capability of generating a sequence that converges at a rate that does not depend on the size of the original system.

A zonal multi-domain approach can better account for multiple-scale solutions such as those occurring in highly structured flows in fluid dynamics, or in

fractured materials in structural mechanics.

Furthermore, domain decomposition can easily encompass the use of different numerical schemes within different subdomains, accounting for diverse behaviours of the physical solution. Even further, they allow the use of different kinds of equations in different subdomains whenever the physics behind the problem has a variable nature therein. This is the case, for example, for viscous–inviscid flow interactions in boundary layers, or the coexistence of heterogeneous materials (insulator and conductor) in electromagnetism.

In this book we illustrate the basic mathematical concepts behind domain decomposition. For any given partial differential equation we derive its multidomain formulation from the analysis of transmission conditions at subdomain interfaces. The relationship with the Steklov–Poincaré problem at subdomain interfaces is illustrated.

Concerning the finite dimensional approximation of the problems that are treated in this book, we confine ourselves to finite elements, purely for the sake of exposition. As a matter of fact, most of our analysis can be applied to any family of Galerkin approximation, such as, for example, spectral element methods or the h–p version of finite elements.

A large variety of boundary value problems is addressed, including symmetric elliptic equations, advection–diffusion equations, the elasticity problem, the Stokes problem for incompressible and compressible fluids, the time-harmonic Maxwell equations, parabolic and hyperbolic equations, and suitable couplings of heterogeneous equations.

We consider both overlapping and non-overlapping subdomain decompositions (although we pay more attention to the latter), and analyse the convergence of several iterative procedures among subdomains.

The reader may feel that this book is mainly confined to a simple decomposition in two subdomains. Actually, this has been a deliberate choice, stemming from the consideration that difficult concepts can be more easily addressed in this context. Moreover, this simple partition is suitable for carrying out the sensitivity of domain decomposition algorithms with respect to the grid refinement. The scalable property of the algorithms in terms of the number of subdomains requires instead an ad hoc analysis, which we sketch for elliptic problems only.

We develop the algebraic part of the algorithms, but mostly focus on the differential interpretation of the numerical methods that we propose. In other words, our attention focuses more on the solvers than on the preconditioners. Indeed, this viewpoint is more suitable for envisaging how to extend these methods to new types of equations and even to heterogeneous situations such as those described earlier.

An outline of this book is sketched below. In Chapter 1 we present the mathematical foundation of domain decomposition methods for both overlapping and non-overlapping domain partitions, in the case of symmetric elliptic equations. Also, we illustrate the relationship with the Steklov–Poincaré problem at subdomains interfaces.

In Chapter 2 we consider the finite element approximation of the Steklov–Poincaré interface problem, as well as its relation with the Schur complement matrix. A brief presentation of non-conforming domain decomposition methods is also included.

In Chapter 3 the finite element approximation of the domain decomposition algorithms presented in Chapter 1 is addressed. In particular, we introduce many preconditioners for the Schur complement matrix based on a substructuring strategy.

The main theoretical convergence results are contained in Chapter 4. First of all, we focus on three extension theorems, which prepare the basis for the analysis of the convergence of substructuring iterative methods. Then we give some abstract convergence theorems for Krylov-type iterations, which are used for showing the convergence of Dirichlet–Neumann and Neumann–Neumann algorithms. The analysis of Robin and Schwarz domain decomposition methods is also presented.

Chapter 5 is devoted to the formulation and analysis of domain decomposition methods for other boundary value problems; precisely, non-symmetric elliptic problems, the linear elasticity problem, the Stokes problem (for both incompressible and compressible flows), the first-order advection problem, and the time-harmonic Maxwell problem.

Some domain decomposition methods specifically suited for advection–diffusion equations are presented and analysed in Chapter 6.

Time-dependent problems are addressed in Chapter 7, for both parabolic and hyperbolic operators. In particular, we evaluate the impact of both implicit and explicit time-advancing finite difference schemes on domain partitioning in space. Non-linear evolution problems, especially the Navier–Stokes and Euler equations in fluid dynamics, are also considered.

Finally, in Chapter 8 we describe several types of heterogeneous domain decomposition methods, which are of particular interest in fluid dynamics and electromagnetism.

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Alfio Quarteroni
Alberto Valli

CONTENTS

1

THE MATHEMATICAL FOUNDATION OF DOMAIN DECOMPOSITION METHODS

In this chapter the reader is encouraged to discover the mathematical foundation of domain decomposition methods, which are based on partitions of the computational domain into subdomains with or without overlap.

We introduce the concept of transmission conditions at subdomain interfaces and the Steklov–Poincaré problem for the interface variables. Both differential and variational formulations are addressed.

Then we present substructuring iterative methods for disjoint subdomains, and the Schwarz alternating method for overlapping subdomains. The convergence analysis of these methods will be carried out in Chapter 4. We also comment on two other approaches: the fictitious domain method and the so-called three-field method.

We deal mainly with *symmetric* linear elliptic boundary value problems, and, in particular, with the Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, and in the rest of this book, Ω is a d-dimensional domain ($d = 2, 3$), with a Lipschitz boundary $\partial\Omega$, whose outer unit normal direction is denoted by $\mathbf{n}^*$, f is a given function of $L^2(\Omega)$, $\Delta := \sum_{j=1}^{d} D_j D_j$ is the Laplace operator and D_j denotes the partial derivative with respect to x_j, $j = 1, \ldots, d$. To start

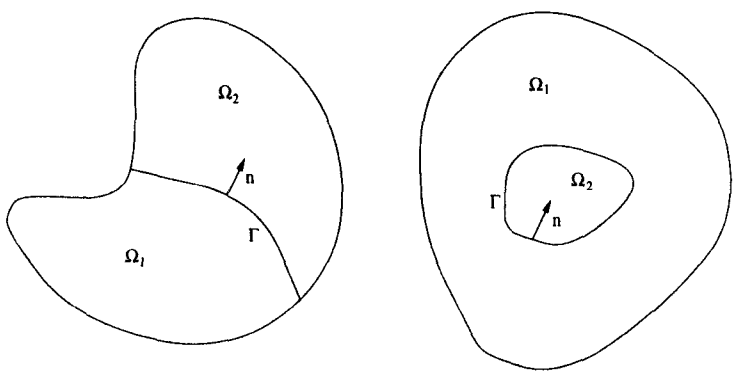

FIG. 1.1. Non-overlapping partition of the domain Ω into two subdomains.

with, we assume that Ω is partitioned into two non-overlapping subdomains Ω_1 and Ω_2, and denote by $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$ (see Fig. 1.1). We also assume that Γ is a Lipschitz $(d-1)$-dimensional manifold.

The generalisation to other boundary value problems will be done in later chapters, particularly Chapters 5, 6 and 8.

The finite dimensional approximation is addressed in the next two chapters.

1.1 Multi-domain formulation and the Steklov–Poincaré interface equation

We indicate by u_i the restriction to Ω_i, $i = 1, 2$, of the solution u to (1.1), and by $\mathbf{n}^i$ the normal direction on $\partial\Omega_i \cap \Gamma$, oriented outward. For simplicity of notation we also set $\mathbf{n} = \mathbf{n}^1$.

It is easily seen that the Poisson problem (1.1) can be reformulated in the equivalent multi-domain form:

$$
\text{(1.1.1)} \qquad \begin{cases} -\Delta u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ \dfrac{\partial u_2}{\partial n} = \dfrac{\partial u_1}{\partial n} & \text{on } \Gamma \\ u_2 = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ -\Delta u_2 = f & \text{in } \Omega_2. \end{cases}
$$

Equations $(1.1.1)_3$ and $(1.1.1)_4$ are the *transmission conditions* for u_1 and u_2 on Γ.

The physical meaning of this split formulation is clear as soon as the original solution of problem (1.1) is smooth enough (say, $u \in C^1(\overline{\Omega})$). In a more general framework the equivalence between (1.1) and (1.1.1) is shown in the next section by resorting to the weak formulation of both problems.

We will see in Section 1.3 that domain decomposition methods are generally amenable to iterative procedures for an *interface* equation that is associated with the given differential problem. This interface problem can be defined in terms of the Steklov–Poincaré operator that we are going to introduce.

Let us refer to the model problem (1.1) and its multi-domain formulation (1.1.1), which corresponds to the domain partition of Fig. 1.1. The same arguments apply to the other boundary value problems that will be described in Section 1.4 (for further details see also Quarteroni and Valli 1991*a*, *b*).

Let λ denote the unknown value of u on Γ. We consider the two Dirichlet problems:

$$\text{(1.1.2)} \qquad \begin{cases} -\Delta w_i = f & \text{in } \Omega_i \\ w_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ w_i = \lambda & \text{on } \Gamma, \end{cases}$$

for $i = 1, 2$. We can obviously state that

$$\text{(1.1.3)} \qquad w_i = u_i^0 + u_i^*,$$

where we have defined u_i^0 and u_i^* to be the solutions of the following Dirichlet problems:

$$\text{(1.1.4)} \qquad \begin{cases} -\Delta u_i^0 = 0 & \text{in } \Omega_i \\ u_i^0 = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ u_i^0 = \lambda & \text{on } \Gamma, \end{cases}$$

and

$$\text{(1.1.5)} \qquad \begin{cases} -\Delta u_i^* = f & \text{in } \Omega_i \\ u_i^* = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ u_i^* = 0 & \text{on } \Gamma. \end{cases}$$

For each $i = 1, 2$, u_i^0 is the *harmonic extension* of λ into Ω_i, and will be denoted $H_i\lambda$. We will also write $\mathcal{G}_i f$ instead of u_i^*.

If we proceed formally, comparing (1.1.1) with (1.1.2), it follows that

$$\text{(1.1.6)} \qquad w_i = u_i \text{ for } i = 1, 2 \text{ if and only if } \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n} \text{ on } \Gamma.$$

The latter condition amounts to the requirement that λ satisfies the *Steklov–Poincaré interface equation*

$$\text{(1.1.7)} \qquad S\lambda = \chi \qquad \text{on } \Gamma,$$

where

$$\text{(1.1.8)} \qquad \begin{aligned} \chi :&= \frac{\partial}{\partial n}\mathcal{G}_2 f - \frac{\partial}{\partial n}\mathcal{G}_1 f \\ &= -\sum_{i=1}^{2} \frac{\partial}{\partial n^i}\mathcal{G}_i f \end{aligned}$$

and S is the *Steklov–Poincaré operator*, which is formally defined as

$$
\begin{aligned}
S\eta :&= \frac{\partial}{\partial n} H_1\eta - \frac{\partial}{\partial n} H_2\eta \\
&= \sum_{i=1}^{2} \frac{\partial}{\partial n^i} H_i\eta.
\end{aligned} \tag{1.1.9}
$$

In particular, S can be split as $S = S_1 + S_2$, with

$$
S_i\eta := \frac{\partial}{\partial n^i} H_i\eta, \quad i = 1, 2. \tag{1.1.10}
$$

This operator, which was introduced a century ago (1896–1900), has been more recently analysed by Agoshkov and Lebedev (1985) in the framework of iterative methods. It is, however, worthwhile to point out that Agoshkov and Lebedev in fact considered the inverse operators S_1^{-1} and S_2^{-1}, and called them Poincaré–Steklov operators.

Remark 1.1.1 (Generalisation) The analysis that we are going to carry out on the Poisson equation will be applied to a far more general differential problem of the form

$$
\mathcal{L}u = f \quad \text{in } \Omega, \tag{1.1.11}
$$

where $\mathcal{L}$ is a partial differential operator, f is a given datum, and u is the unknown solution. A broad range of problems can be considered (see Chapter 5), including non-symmetric elliptic problems, the linear elasticity problem, the Stokes problem for incompressible flows, the viscous and inviscid Stokes problem for compressible flows, and the time-harmonic Maxwell system.

Should Ω be partitioned into two disjoint subdomains Ω_1 and Ω_2 as indicated in Fig. 1.1, we can go along the same lines presented above to generate a split version of problem (1.1.11). Denoting again for $i = 1, 2$ by u_i the restriction of u to Ω_i, it follows from (1.1.11) that

$$
\begin{aligned}
\mathcal{L}u_1 &= f \quad \text{in } \Omega_1 \\
\mathcal{L}u_2 &= f \quad \text{in } \Omega_2.
\end{aligned} \tag{1.1.12}
$$

To guarantee the equivalence with (1.1.11) we need to enforce transmission conditions between u_1 and u_2 across Γ. In an abstract form, such conditions can be expressed by the two relationships

$$
\begin{aligned}
\Phi(u_1) &= \Phi(u_2) \text{ on } \Gamma \\
\Psi(u_1) &= \Psi(u_2) \text{ on } \Gamma,
\end{aligned} \tag{1.1.13}
$$

where the functions Φ and Ψ will depend upon the nature of the problem.

Typically, for second-order elliptic operators, (1.1.13) expresses the continuity across Γ of u and of the normal 'flux' (namely, the normal stress) involving first-order derivatives of u_1 and u_2. More generally, these interface conditions are most often determined noting that:

- *The solution u belongs to a space of functions defined over the whole Ω. This requires that $u_{|\Omega_1}$ in Ω_1 and $u_{|\Omega_2}$ in Ω_2 enjoy a certain regularity therein, and in addition that they satisfy a suitable matching on Γ.*
- *The restrictions $u_{|\Omega_1}$ and $u_{|\Omega_2}$ are distributional solutions to the given equation in Ω_1 and Ω_2, respectively. Another interface condition between them comes from the fact that u in fact satisfies the equation in the sense of distributions in the whole Ω; namely, through the interface Γ and not only separately in Ω_1 and Ω_2.*

For the Poisson problem (1.1.1) the obvious identification

$$\Phi(v) = v, \quad \Psi(v) = \frac{\partial v}{\partial n}$$

holds true. Keeping in mind this correspondence, all iterative substructuring methods that we are going to introduce for the Poisson problem in Sections 1.3, 1.4 can actually be extended to the more general problems (1.1.12), (1.1.13) in a straightforward manner. This is the case, in particular, for the classical methods like Dirichlet–Neumann, Neumann–Neumann or Robin, originally introduced for the Laplace operator, and here applied to a very general family of boundary value problems (see Chapter 5). □

Remark 1.1.2 In transforming the Dirichlet boundary value problem (1.1) into an equation on Γ, we have chosen as interface unknown the *physical* variable that has to match on Γ as described in the first condition in Remark 1.1.1, while the interface equation (1.1.7) for the Steklov–Poincaré operator S is based on ensuring that the second condition is satisfied. The same procedure will be constantly followed in the remainder of this book, but clearly other choices of interface equation could be devised. These are amenable to different forms of the Steklov–Poincaré operator, and, correspondingly, different iterative substructuring methods. □

1.2 Variational formulation of the multi-domain problem

In this section we formulate (1.1) in a variational way. This requires us to introduce Sobolev spaces and to take into account some of their properties. We will not dwell here on this argument, and refer the interested reader to the comprehensive presentation of this theory that can be found, for example, in J.-L. Lions and Magenes (1972) (see also Chapter 9).

By integrating by parts in Ω, it is easily seen that the weak formulation of (1.1) reads

$$\text{find } u \in V \ : \ a(u,v) = (f,v) \qquad \forall\, v \in V, \tag{1.2.1}$$

where

$$
\begin{aligned}
(w,v) &:= \int_\Omega w\,v \\
a(w,v) &:= (\nabla w, \nabla v) \\
H^1(\Omega) &:= \{v \in L^2(\Omega) \,|\, D_j v \in L^2(\Omega),\ j=1,\ldots,d\} \\
H_0^1(\Omega) &:= \{v \in H^1(\Omega) \,|\, v_{|\partial\Omega} = 0\} \\
V &:= H_0^1(\Omega)
\end{aligned}
$$

and $v_{|\partial\Omega}$ denotes the trace of v (that is, its restriction) on $\partial\Omega$. The norm of $H^1(\Omega)$ will be denoted by $\|\cdot\|_{1,\Omega}$, while $\|\cdot\|_{0,\Omega}$ will indicate the norm of $L^2(\Omega)$. We recall that

$$
\|v\|_{0,\Omega} = (v,v)^{1/2},
$$

while

$$
\|v\|_{1,\Omega} = \left(\|v\|_{0,\Omega}^2 + \sum_{j=1}^{d} \|D_j v\|_{0,\Omega}^2 \right)^{1/2}
$$

for each $v \in H^1(\Omega)$.

The Poincaré inequality states that there exists a constant $C_\Omega > 0$ such that

$$
\int_\Omega v^2 \le C_\Omega \int_\Omega \sum_{j=1}^{d} (D_j v)^2 \qquad \forall\, v \in H_0^1(\Omega). \tag{1.2.2}
$$

Therefore, the norm $\|v\|_{1,\Omega}$ is equivalent to the norm $\|\nabla v\|_{0,\Omega}$ for each $v \in H_0^1(\Omega)$. It is worthwhile to note that the same result is true for functions that vanish only on an open and non-empty subset Σ of $\partial\Omega$.

We also recall that the trace space of $H^1(\Omega)$ on the boundary $\partial\Omega$ is denoted $H^{1/2}(\partial\Omega)$. In an analogous way, the trace space on an open and non-empty subset $\Sigma \subset \partial\Omega$ is indicated by $H^{1/2}(\Sigma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is surjective and continuous; that is, the following trace inequality holds

$$
\|v_{|\partial\Omega}\|_{1/2,\partial\Omega} \le C_\Omega^* \|v\|_{1,\Omega} \qquad \forall\, v \in H^1(\Omega), \tag{1.2.3}
$$

where $\|\cdot\|_{1/2,\partial\Omega}$ denotes the norm in $H^{1/2}(\partial\Omega)$. Moreover, it can be shown that there exist injective, linear, and continuous extension operators from $H^{1/2}(\partial\Omega)$ to $H^1(\Omega)$.

Let us also consider the weak multi-domain formulation equivalent to (1.2.1). First of all, let us set

$$
\begin{aligned}
(w_i, v_i)_{\Omega_i} &:= \int_{\Omega_i} w_i\, v_i \\
a_i(w_i, v_i) &:= (\nabla w_i, \nabla v_i)_{\Omega_i} \\
V_i &:= \{v_i \in H^1(\Omega_i) \,|\, v_{i|\partial\Omega\cap\partial\Omega_i} = 0\} \\
V_i^0 &:= H_0^1(\Omega_i) \\
\Lambda &:= \{\eta \in H^{1/2}(\Gamma) \,|\, \eta = v_{|\Gamma} \text{ for a suitable } v \in V\}.
\end{aligned} \tag{1.2.4}
$$