Gareth A. Jones and J. Mary Jones

# Elementary Number Theory

# 基本数论

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# Preface

Our intention in writing this book is to give an elementary introduction to number theory which does not demand a great deal of mathematical background or maturity from the reader, and which can be read and understood with no extra assistance. Our first three chapters are based almost entirely on A-level mathematics, while the next five require little else beyond some elementary group theory. It is only in the last three chapters, where we treat more advanced topics, including recent developments, that we require greater mathematical background; here we use some basic ideas which students would expect to meet in the first year or so of a typical undergraduate course in mathematics. Throughout the book, we have attempted to explain our arguments as fully and as clearly as possible, with plenty of worked examples and with outline solutions for all the exercises.

There are several good reasons for choosing number theory as a subject. It has a long and interesting history, ranging from the earliest recorded times to the present day (see Chapter 11, for instance, on Fermat's Last Theorem), and its problems have attracted many of the greatest mathematicians; consequently the study of number theory is an excellent introduction to the development and achievements of mathematics (and, indeed, some of its failures). In particular, the explicit nature of many of its problems, concerning basic properties of integers, makes number theory a particularly suitable subject in which to present modern mathematics in elementary terms.

A second reason is that many students nowadays are unfamiliar with the notion of formal proof; this is best taught in a concrete setting, rather than as an abstract exercise in logic, but earlier choices of context, such as geometry and analysis, have suffered from the conceptual difficulty and abstract nature of their subject-matter, whereas number theory is about very familiar and easily manipulated objects, namely integers. We therefore see this book as a vehicle for

explaining how mathematicians go about their business, finding experimental evidence, making conjectures, creating proofs and counterexamples, and so on.

A third reason is that many students prefer computation to abstraction, and number theory, with its discrete, precise nature, is an ideal topic in which to perform numerical experiments and calculations. Many of these can be done by hand, and throughout the book we have given examples and exercises of an algorithmic nature. Nowadays, almost every student has access to computing facilities far in excess of anything the great calculator Gauss could have imagined, and for a few of our exercises such electronic assistance is desirable or even essential. We have not linked our approach to any particular machine, programming language or computer algebra system, since even a fairly primitive pocket calculator or personal computer can greatly enhance one's ability to do number theory (and part of the fun lies in persuading it to do so).

A final reason for learning number theory is that, despite Hardy's (1940) famous but now out-dated claim, it is useful. Its best-known modern application is to the cryptographic systems which allow banks, commercial companies, military establishments, and so on to exchange information in securely-encoded form; many of these systems are based on such number-theoretic properties as the apparent difficulty of factorising very large integers (see Chapters 2 and 5). Physicists, engineers and computer scientists are also finding that number-theoretic concepts are playing an increasing role in their work. These applications were not the original motivation for the great developments in number theory, but their emergence can only add to the importance of the subject.

The first three chapters of this book are intended to be accessible to anyone with a little A-level mathematics. In particular, they are suitable for first-year university students and for the more advanced sixth-formers. Equivalence relations appear in Chapter 3, but otherwise no abstract mathematics is used. Proof by induction is used several times, and three versions of this (including strong induction and the well-ordering principle) are summarised in Appendix A. Chapters 4–8 are a little more algebraic in flavour, and require slightly greater mathematical maturity. Here, it is helpful if the reader has met some elementary group theory (subgroups, cyclic groups, direct products, isomorphisms), and knows what rings and fields are; these topics are summarised in Appendix B. Probabilities are also mentioned, though not in any essential way. These chapters are therefore suitable for second- or third-year students, and also for those first-year students sufficiently interested to want to read further. The last three chapters are more advanced, relying on ideas from other areas of mathematics such as analysis, calculus, geometry and algebra which students will almost certainly have met early in their undergraduate studies; these include convergence (summarised in Appendix C), power series, complex numbers and vector spaces. These chapters should therefore be suitable for

students at second- or third-year level. The final chapter, which traces Fermat's Last Theorem from its ancient roots to its recent proof, is rather more descriptive and historical in style than the others, but we have tried to include sufficient technical detail to give the reader a flavour of this exciting topic.

The early parts of the book could be used as a first-year introduction to the concepts and methods of pure mathematics, while the rest could form the basis for a more specialised second- or third-year course in number theory. Indeed, many of the chapters are based on courses we have taught to first- and third-year mathematics students at the University of Southampton. The book is also suitable for other students, such as computer scientists and physicists, who want an elementary introduction which brings them up to date with recent developments in the subject.

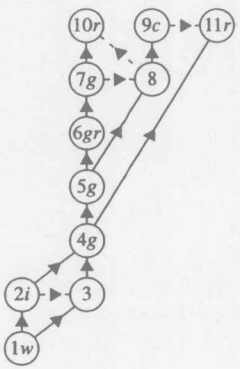
The two essentials for starting number theory are confidence with traditional algebraic manipulation, and some conception of formal proof. Unfortunately, the recent expansion of university education in the UK has coincided with a decline in numbers taking Further Mathematics A-level, so mathematics students now arrive at university much less familiar with these topics than their predecessors were. In our first few chapters we have therefore taken a more leisurely approach than is traditional, using simple results in number theory to illustrate methods of proof, and emphasising algorithmic and computational aspects in parallel with theory. In later chapters, the pace is rather brisker, but even here we have attempted to present our arguments in as simple terms as possible in order to make them more widely accessible. In the case of some advanced results, this has forced us to concentrate on special cases, or to give only outline proofs, but we think this is a worthwhile sacrifice if it conveys to our readers some feeling of what high-level mathematics is like and how it is done – too many mathematics students graduate with only the vaguest idea of the great problems and achievements of their subject.

We would like to thank Peter Neumann for showing us how to discover and communicate mathematics, and many of our colleagues at Southampton, especially Ann and Keith Hirst and David Singerman, for their sound advice on teaching mathematics in general and number theory in particular. We are very grateful to Susan Hezlet and her colleagues at Springer for their advice and encouragement. It is also traditional to thank one's partner for patience and tolerance during the preparation of a book; instead, we shall simply thank our children for not playing their music any louder than was absolutely necessary.

# Notes to the Reader

Mathematics is a difficult subject to read, and number theory is no exception, even if its subject matter is less abstract than some other topics. Do not be surprised, therefore, if it takes you several attempts before you completely understand an argument. It is often useful when reading mathematics to make notes and to do calculations as you go along; for instance, a general argument can often be clarified by seeing how it works in some specific cases.

Exercises are an important part of the learning process, and you are encouraged to attempt them while reading each section; we have generally placed them immediately after the topics on which they are based, to reinforce your understanding of those topics. Supplementary exercises, which are generally more demanding, are placed at the end of a chapter; they can refer to anything in that chapter, and possibly also to topics covered in earlier chapters. Answers or outline solutions for all the exercises are given at the end of the book; how-



ever, there is a great deal more to be gained from trying the exercises first, before reading the solutions!

The diagram on page xiii shows the interdependence of chapters, with continuous and broken lines indicating strong and weak links. Thus, to understand Chapter 11 it is sufficient to have read Chapters 1–4, though it also helps to know a little of the material in Chapter 9. The letters $i$ and $w$ indicate that the principles of induction and well-ordering are used; these are summarised in Appendix A. Similarly $g$ and $r$ refer to material on groups and rings (Appendix B), and $c$ to convergence (Appendix C).

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# Contents

# 1
# Divisibility

We start with a number of fairly elementary results and techniques, mainly about greatest common divisors. You have probably met some of this material already, though it may not have been treated as formally as here. There are several good reasons for giving very precise definitions and proofs, even when there is general agreement about the validity of the mathematics involved. The first is that 'general agreement' is not the same as convincing proof: it is not unknown for majority opinion to be seriously mistaken about some point. A second reason is that, if we know exactly what assumptions are required in order to deduce certain conclusions, then we may be able to deduce similar conclusions in other areas where the same assumptions hold true. For example, this chapter is entirely devoted to the divisibility properties of *integers*, but it turns out that very similar definitions, methods and theorems are valid for certain other objects which can be added, subtracted and multiplied; some of these objects, such as polynomials, are very familiar, while others, such as Gaussian integers and quaternions, will be introduced in later chapters. These generalisations of the integers are also explored in algebra, under the heading of ring theory.

## 1.1 Divisors

Our starting-point is the *division algorithm*, which is as follows:

### Theorem 1.1

If $a$ and $b$ are integers with $b > 0$, then there is a unique pair of integers $q$ and $r$ such that

$$a = qb + r \qquad \text{and} \qquad 0 \leq r < b\,.$$

### Example 1.1

If $a = 9$ and $b = 4$ then we have $9 = 2 \times 4 + 1$ with $0 \leq 1 < 4$, so $q = 2$ and $r = 1$; if $a = -9$ and $b = 4$ then $q = -3$ and $r = 3$.

In Theorem 1.1, we call $q$ the *quotient* and $r$ the *remainder*. By dividing by $b$, so that

$$\frac{a}{b} = q + \frac{r}{b} \qquad \text{and} \qquad 0 \leq \frac{r}{b} < 1\,,$$

we see that $q$ is the integer part $\lfloor a/b \rfloor$ of $a/b$, the greatest integer $i \leq a/b$. This makes it easy to calculate $q$, and then to find $r = a - qb$.

### Proof

First we prove existence. Let

$$S = \{a - nb \mid n \in \mathbb{Z}\} = \{\, a, a \pm b, a \pm 2b, \ldots \}\,.$$

This set of integers contains non-negative elements (take $n = -|a|$), so $S \cap \mathbb{N}$ is a non-empty subset of $\mathbb{N}$; by the well-ordering principle (see Appendix A), $S \cap \mathbb{N}$ has a least element, which has the form $r = a - qb \geq 0$ for some integer $q$. Thus $a = qb + r$ with $r \geq 0$. If $r \geq b$ then $S$ contains a non-negative element $a - (q+1)b = r - b < r$; this contradicts the minimality of $r$, so we must have $r < b$.

To prove uniqueness, suppose that $a = qb + r = q'b + r'$ with $0 \leq r < b$ and $0 \leq r' < b$, so $r - r' = (q' - q)b$. If $q' \neq q$ then $|q' - q| \geq 1$, so $|r - r'| \geq |b| = b$, which is impossible since $r$ and $r'$ lie between 0 and $b-1$ inclusive. Hence $q' = q$ and so $r' = r$. □

We can now deal with the case $b < 0$: since $-b > 0$, Theorem 1.1 implies that there exist integers $q^*$ and $r$ such that $a = q^*(-b) + r$ and $0 \leq r < -b$, so

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