FUNCTIONS OF TWO VARIABLES

SEÁN DINEEN

CHAPMAN & HALL MATHEMATICS

Functions of Two Variables

Seán Dineen

Professor of Mathematics
Department of Mathematics
University College Dublin
Ireland

CHAPMAN & HALL
London · Glasgow · Weinheim · New York · Tokyo · Melbourne · Madras

Published by
Chapman & Hall, 2–6 Boundary Row, London SE1 8HN, UK

Chapman & Hall, 2–6 Boundary Row, London SE1 8HN, UK

Blackie Academic & Professional, Wester Cleddens Road, Bishopbriggs, Glasgow G64 2NZ, UK

Chapman & Hall GmbH, Pappelallee 3, 69469 Weinheim, Germany

Chapman & Hall USA, 115 Fifth Avenue, New York, NY 10003, USA

Chapman & Hall Japan, ITP-Japan, Kyowa Building, 3F, 2-2-1 Hirakawacho, Chiyoda-ku, Tokyo 102, Japan

Chapman & Hall Australia, 102 Dodds Street, South Melbourne, Victoria 3205, Australia

Chapman & Hall India, R. Seshadri, 32 Second Main Road, CIT East, Madras 600 035, India

First edition 1995

Printed in Great Britain by St Edmundsbury Press, Suffolk

ISBN 0 412 70760 8

A catalogue record for this book is available from the British Library

∞ Printed on permanent acid-free text paper, manufactured in accordance with ANSI/NISO Z39.48-1992 and ANSI/NISO Z39.48-1984 (Permanence on Paper).

CHAPMAN & HALL MATHEMATICS SERIES

OTHER TITLES IN THE SERIES INCLUDE

Functions of Two Variables
S. Dineen

Network Optimization
V.K. Balakrishnan

Sets, Functions and Logic
A foundation course in mathematics
Second edition
K. Devlin

Algebraic Numbers and Algebraic Functions
P.M. Cohn

Dynamical Systems
Differential equations, maps and chaotic behaviour
D.K. Arrowsmith and C.M. Place

Control and Optimization
B.D. Craven

Elements of Linear Algebra
P.M. Cohn

The Dynamic Cosmos
M. Madsen

Full information on the complete range of Chapman & Hall mathematics books is available from the publishers.

To SARA

who likes mathematics and little books

Preface

This book was initially based on a short course of twenty lectures, given to second year students at University College Dublin during the autumn of 1992. Later, two chapters on integration theory were added to improve the balance between differential and integral calculus. The students had completed a one year course on differential and integral calculus for real valued functions of one real variable—this is the prerequisite for reading this book—and this course was designed as an introduction to the calculus of several variables.

My initial motivation for writing this book was to provide my own students with a friendly set of notes that they could read in their entirety. As the book took shape, I realized that I was addressing the subject in a manner somewhat different to the standard texts on several variable calculus. It is difficult to explain precisely why this occurred. Nevertheless, an attempted explanation may also help you, the reader, in your approach and I will try to give a partial one.

Research mathematicians typically spend their working lives doing research, learning new mathematics and teaching. They teach themselves new mathematics mainly to further their own research. Yet, often their own way of learning mathematics is the complete opposite to the way they present mathematics to their students. On approaching a new area of mathematics the research mathematician is usually looking for some result (or technique). He or she will generally not know precisely what is being sought and lives in hope that by searching, often backwards and forwards through a text, the required result will somehow be recognized. The search through the literature will neither be random nor logical but will be based on accumulated experience and intuition. Once the objective has been identified the research mathematician works backwards to satisfy professional standards for a precise meaning of the terms involved and the context in which the result may be applied. Finally, and this depends on many things, the research mathematician may even decide to satisfy a need for certainty and will then work through the background proofs. Thus the mathematician, when doing research, behaves like a detective and in fact there is no alternative since the plot is not revealed until the story is almost over. Nevertheless, with students we first reveal the climax (theorem), then the evidence (proof) and finally the intuition (explanation and examples). This robs the subject of its excitement and does not use the students' own intuition and experience. I have tried to approach the material of these lectures as a research mathematician approaches research: full of doubt, more intuitively than logically, somewhat imprecise about where we may be going, but with a general objective in mind, moving backwards and forwards, trying simple cases, using various tricks that have previously proved useful, experimenting and eventually arriving at something of interest. Having obtained useful results intuitively I have returned to justify them mathematically. At this stage the reasoning behind the proofs is often more acceptable and the proofs themselves become an integral part of a unified process by adding to our understanding of the applications, by showing the usefulness of earlier theoretical results and by suggesting further developments. Of course, I have not fully succeeded in this attempt, but feel nevertheless that I have gone some way in this direction. I believe that this is almost the only way to learn mathematics and that most students are trying to follow this approach.

Although the calculus of several variables is often presented as a fully mature subject in its own right, it is clear that most of the concepts are the natural evolution of attempting to imitate the one dimensional theory and I have tried to follow this approach in my presentation. The restriction to functions of two variables simplifies the notation and at the same time introduces most of the main concepts that arise in higher dimensions. I believe that a clear understanding of the two variables case is a suitable introduction to the higher dimensional situation. I have tried to be both rigorous and self-contained and so have clearly marked out assumptions made and discussed the significance of results used without proof.

We discuss all possible functions which involve two variables and so look at functions from $\mathbf{R}^2 \to \mathbf{R}$, $\mathbf{R} \to \mathbf{R}^2$ and $\mathbf{R}^2 \to \mathbf{R}^2$. This provides a basic introduction to three subjects, i.e. calculus of several variables, differential geometry and complex analysis.

In the first twelve chapters we discuss maxima and minima of functions of two variables on both open sets and level curves. Second order derivatives and the Hessian are used to find extremal values on open sets while the method of Lagrange multipliers is developed for level curves. In the process we introduce partial derivatives, directional derivatives, the gradient, critical points, tangent planes, the normal line and the chain rule and also discuss regularity conditions such as continuity of functions and their partial derivatives and the relationship between differentiation and approximation. In chapters 13 to 16 we investigate the curvature of plane curves. Chapters 18 to 22 are devoted to integration theory on $\mathbf{R}^2$. We study Fubini's theorem (on the change of order of integration), line and area integrals and connect them using Green's theorem. In chapter 17 we introduce holomorphic (or C-differentiable) functions and using approximation methods derive the Cauchy-Riemann equations. This introduction to complex analysis is augmented in the final chapter where Green's theorem is combined with the Cauchy-Riemann equations to prove Cauchy's theorem. Partial derivatives enter into and play an important role in every topic discussed.

As life is not simple many things depend on more than one variable and it is thus not surprising that the methods developed in this book are widely used in the physical sciences, economics, statistics and engineering. We mention some of these applications and give a number of examples in the text.

Anyone interested in several variable calculus will profit from reading this book. Students suddenly exposed to the multidimensional situation in its full generality will find a gentle introduction here. Students of engineering, economics and science who ask simple but fundamental questions will find some answers and, perhaps, even more questions here.

This book may be used fully or partially as the basis for a course in which a lecturer has the option of inserting extra material and developing more fully certain topics. Alternatively it can be used as supplementary reading for courses on advanced calculus or for self study. The material covered in each chapter can be presented in approximately sixty minutes, although in some of my lectures I was not able to cover fully all examples in the allocated time, and for aesthetical and mathematical reasons I have, in writing up these lectures, sometimes ignored the time frame imposed by the classroom.

It is a real pleasure to acknowledge the help I have received in bringing this project to fruition. Siobhán Purcell displayed patience, understanding and skill in preparing the written text. Alun Carr, Peter O'Neill and Brendan Quigley gave valuable advice on preparing the diagrams, while William Aliaga-Kelly, Derek O'Connor, Michael Mackey and Richard Timoney performed the onorous task of producing the diagrams. Maciej Klimek my former colleague (now at Uppsala), Milne Anderson (London) and Pauline Mellon (Dublin) provided valuable advice and sustained support at all stages. The mathematics editor at Chapman & Hall, Achi Dosanjh, gave encouragement and practical support at crucial times. To all these and to my students at University College Dublin, I offer sincere thanks. I hope you, the reader, will enjoy this book and I welcome your comments. I can be contacted at the address below.

Department of Mathematics,
University College Dublin,
Belfield, Dublin 4 (Ireland)

Contents

1

Functions from $\mathbf{R}^2$ to $\mathbf{R}$

Summary. *We introduce the problem of finding the maximum and minimum of a real valued function of two variables. The one dimensional theory suggests that we discuss the problem for functions defined on open sets and their boundaries. We define open sets, consider a possible definition of the derivative and define the graph of a function of two variables.*

In this course we will discuss all possible functions which involve two variables and so look at functions from $\mathbf{R}^2$ into $\mathbf{R}$, from $\mathbf{R}$ into $\mathbf{R}^2$ and from $\mathbf{R}^2$ into $\mathbf{R}^2$. We begin by considering functions from $\mathbf{R}^2$ into $\mathbf{R}$ and our objective is to obtain methods for finding maxima and minima. If the functions are arbitrarily behaved we get nowhere, so we have to make some assumptions—we will use the general term **regularity conditions**—on the functions considered. These regularity conditions usually relate to continuity and differentiability. First, however, we try and see, based on our one dimensional experience, how we might proceed and then return to look more closely at what we need in order to proceed. The main one dimensional motivation is the following fundamental result.

Theorem 1. *If $f: [a, b] \longrightarrow \mathbf{R}$ is continuous on the closed interval $[a, b]$ then f has a maximum and a minimum on $[a, b]$.*

In other words there are two points in $[a, b]$, x_1 and x_2, such that

$$f(x_1) \leq f(x) \leq f(x_2)$$

for all x in $[a, b]$.

The function f has a minimum value $f(x_1)$ which is achieved at x_1 and a maximum value $f(x_2)$ which is achieved at x_2 (figure 1.1). Both the maximum and minimum are finite. The maximum (or minimum) may appear in two ways:

(i) it may occur at a point inside $[a, b]$, i.e. in (a, b) or
(ii) it may occur at a boundary point, i.e. at either a or b.

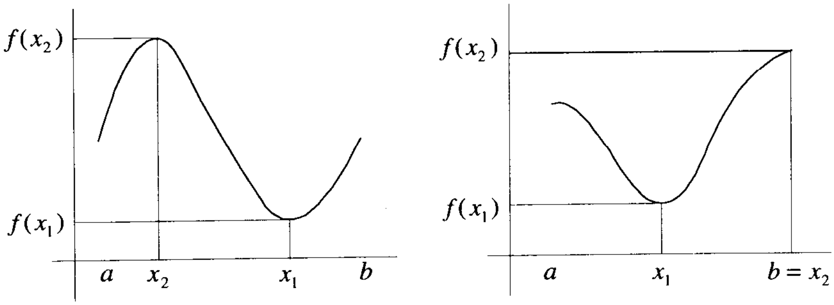

Figure 1.1

On the left in figure 1.1 we see that possibility (i) occurs for both maximum and minimum while on the right possibility (ii) occurs for the maximum while possibility (i) occurs for the minimum.

If f is differentiable on (a, b) and the maximum occurs inside then we have $f'(x_2) = 0$ so our method of proceeding in this case is to look at all x in (a, b) where $f'(x) = 0$. We call these the **critical points** of f. There is usually only a small number of critical points so we can evaluate f at these points. This takes care of all possibilities inside and since there are only two other points—the end points a and b—we can find $f(a)$ and $f(b)$ and locate the maximum by choosing which one of this small set of possibilities gives the largest value of the function.

If we are now considering a function f defined on some subset U of $\mathbf{R}^2$ it is natural to attempt to try and break the problem (of finding the maximum) into two problems. To deal with f "inside" we have to define what the derivative of f might mean.

From the one dimensional theory we see that $f'(x)$ only makes sense if f is defined at all points near x and, indeed, any method of deciding that a point x is a maximum or minimum of the function f must in some way involve the values of f at all points near x. So we first need to define a suitable analogue of an **open interval** in $\mathbf{R}^2$. We are thus led to the following definition.

Definition 2. *A subset U of $\mathbf{R}^2$ is open if for each point (x_0, y_0) in U there exists $\delta > 0$ such that*

$$\{(x, y); (x - x_0)^2 + (y - y_0)^2 < \delta^2\} \subset U.$$

So a set is open if each point in the set can be **surrounded** by a **small disc of positive** radius which still lies in the set.

Example 3. $\{(x,y); x^2+y^2<1\}$ is open (figure 1.2).

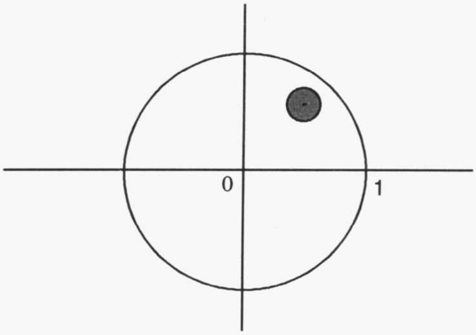

Figure 1.2

Example 4. $\{(x,y); x^2+y^2=1\}$ is not open (figure 1.3).

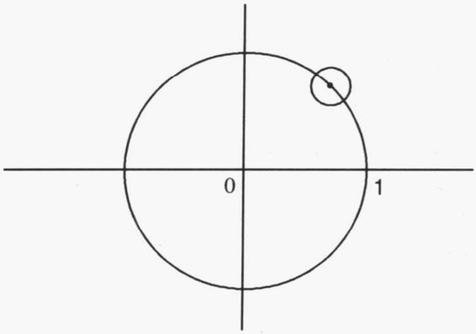

Figure 1.3

Example 5. $\{(x,y); x^2+y^2\le 1\}$ is not open (figure 1.4). Points P inside are all right but points Q on the boundary are not.

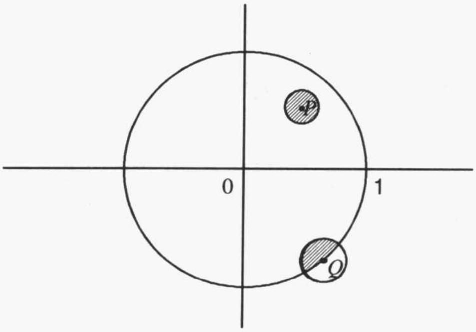

Figure 1.4

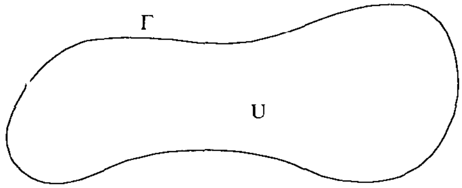

Figure 1.5

Notice that the open set in the above examples has $<$ signs in its definition while the sets which are not open have $\leq$ or $=$. Why?

We also see immediately a new complication in finding the maximum if it occurs on the boundary.

The boundary of $[a, b]$ consisted of just two points whereas the boundary of $U \subset \mathbf{R}^2$ consists of a curve Γ (see figure 1.5)—which contains an infinite number of points—and so the method of evaluating the function at each boundary point is impossible—there are too many points—and a new method has to be devised. We do this later—it is called the method of **Lagrange multipliers**.

So far we have only defined a set on which we might hope to differentiate. Now we will try to define the derivative of f and afterwards try to see what we might mean by a critical point. The task of a critical point is to help locate maxima and minima. From the one variable theory we have

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.$$

We consider the following possibility:

$$\lim_{(x,y) \to (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0)}{(x,y) - (x_0,y_0)}$$
$$= \lim_{(\Delta x,\Delta y) \to (0,0)} \frac{f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0,y_0)}{(\Delta x, \Delta y)}.$$

We immediately run into difficulties. Apart altogether from the possible definitions of limit the expression

$$\frac{f(x,y) - f(x_0,y_0)}{(x,y) - (x_0,y_0)}$$

does not make sense. We cannot divide by a vector. We could identify (x, y) with the complex number $z = x + iy$ and we may then divide by complex numbers. This in fact is interesting when considering functions into $\mathbf{R}^2$ and we will look into it later (chapter 17). However, for real valued functions it would lead to a definition in which the only differentiable functions are the constant functions (see example 62) and this clearly is of no use in finding maxima and minima. Why? Moreover, if we move up one dimension to $\mathbf{R}^3$ how do we divide by $(\Delta x, \Delta y, \Delta z)$? This won't work. We return to fundamentals and try another approach.

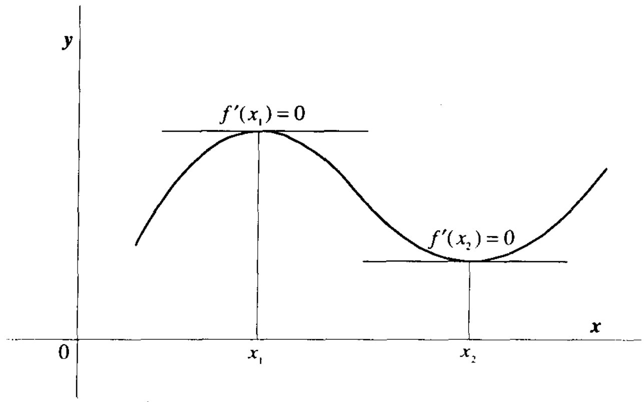

Figure 1.6

In the one dimensional theory we are also led to critical points by considering the **graph** and noting that the line that fits closest to the graph—the **tangent line**—is horizontal at all critical points (figure 1.6) and that in general the slope of this line is the derivative (figure 1.7).

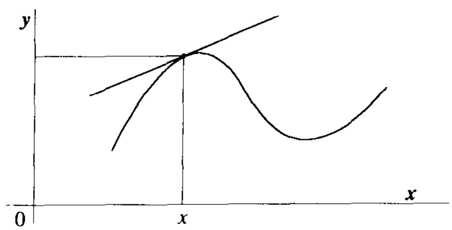

Figure 1.7

We will try and draw the graph to see if it leads to anything of interest.

The graph of the function of one variable $y = f(x)$ consists of all points in $\mathbf{R}^2$ of the form $(x, f(x))$ or (x, y) where $y = f(x)$.

Definition 6. *If $f : U \subset \mathbf{R}^2 \to \mathbf{R}$ then the graph of f consists of the set of all points*

$$\{(x, y, f(x, y)); (x, y) \subset U\}.$$

Since we are already using x and y to denote variables we often let $z = f(x, y)$ and then we are considering the points (x, y, z) where $z = f(x, y)$.

Exercises

1.1 Find the natural domain of definition of the following functions (i.e. find where the functions make sense):

$$\text{(a)} \quad f(x, y) = \frac{y+2}{x} \qquad \text{(b)} \quad f(u, v) = \frac{uv}{u - 2v}$$

$$\text{(c)} \quad f(r, s) = (\log(r/s)) - \sqrt{1 - r}.$$

1.2 Sketch the function $f : \mathbf{R} \to \mathbf{R}$, $f(x) = x^3 - 3x$.
Find, by inspection, intervals $[a, b]$, $[c, d]$ and $[e, f]$ such that

(a) f achieves its maximum and minimum on $[a, b]$ at a and b respectively,

(b) f achieves its maximum on $[c, d]$ both at the point d and at some point in (c, d),

(c) f achieves both its maximum and minimum on $[e, f]$ at precisely two points.

1.3 Let $f(x, y) = \dfrac{x^2}{4} + \dfrac{y^2}{9}$ and $U = \{(x, y); f(x, y) < 1\}$. Sketch the set U. Show that the point $(-1, 2)$ lies in U. Find a positive δ such that

$$\{(x, y); x^2 + (y - 1)^2 < \delta^2\} \subset U.$$

Show that U is open. What formula describes the boundary of U?

1.4 Give an example of an open set in $\mathbf{R}^2$ whose boundary consists of two points.

1.5 Show that the function

$$f(x) = 2x^3 - 9x^2 + 12x$$

has a local maximum and a local minimum, but no maximum and no minimum on the set $(0, 3)$.

1.6 By using one variable calculus find the minimum of the function

$$f(x) = x^2 - 10x + 9, \quad x \in \mathbf{R}.$$

Sketch the graph of f.

1.7 Let $f : \mathbf{R} \to \mathbf{R}$ denote a function with continuous first and second derivatives and suppose $f''(x) \neq 0$ for all $x \in \mathbf{R}$. By drawing sketches convince yourself that f has a minimum on $\mathbf{R}$ in each of the following cases:

(a) f has one local minimum and no local maximum,

(b) f has precisely two local minima and one local maximum.

2

Partial Derivatives

Summary. *We define and examine level sets of the graph and arrive at the concept of the partial derivative. Examples of partial derivatives are given.*

The graph of $f: U \subset \mathbf{R}^2 \to \mathbf{R}$ is a subset of $\mathbf{R}^3$ and we will call it a **surface**. As we get to study graphs we will see that they have many of the features that we intuitively associate with surfaces and so our use of the word surface is not unreasonable. Since the graph is a subset of $\mathbf{R}^3$ a certain amount of ingenuity is required in order to obtain a faithful sketch and a number of standard approaches to this problem have been developed over the years. One method is to consider **cross sections** and by examining sufficiently many cross sections we may get ideas on where the maximum or minimum might be located.

A cross section of $\mathbf{R}^2$ is a **line** and this is determined by a linear equation in two variables $ax + by = c$. A cross section of $\mathbf{R}^3$ is a **plane** and this is determined by a linear equation in three variables

$$ax + by + cz = d$$

where a, b, c and d are real numbers, and by varying these numbers we get different cross sections. A cross section of a surface consists of the points on the surface which satisfy the linear equation.

There are many choices for a, b, c and d. We will follow a general principle in mathematics—take the easiest cases first and examine them. If we are lucky we get what we want. If we are not lucky we at least have some experience when we have to consider more complicated cases. A second principle also comes in here—put as many constants as possible equal to 0 in the first instance and after that take as many as possible equal to 1. I say "as many as possible" since if we are too enthusiastic we end up with a situation which is completely trivial.