Mathematical Modeling in Continuum Mechanics

连续力学中的数学模型

Roger Temam
Alain Miranville

CAMBRIDGE
世界图书出版公司

MATHEMATICAL MODELING IN CONTINUUM MECHANICS

ROGER TEMAM
Université Paris-Sud, Orsay and Indiana University

ALAIN MIRANVILLE
Université de Poitiers

CAMBRIDGE
UNIVERSITY PRESS
世界图书出版公司

Sold and distributed in the People's Republic of China by Science Press Beijing

Sold and distributed all over the world with the exception of the People's Republic of China by International Press

Published by

Science Press
16 Donghuangchenggen North Street
Beijing 100717, China

International Press
P. O. Box 2872
Cambridge, MA 02238-2872
U. S. A

ISBN 7-03-010271-1/O-1599(Science Press)
ISBN 1-57146-125-6(International Press)

Introduction

At a time when mathematical modeling is pervading many areas of science and master's degree programs in industrial mathematics are being initiated in many universities, this book is intended as an introduction to continuum mechanics and mathematical modeling. One of the aims of the book is to reduce the gap slightly between mathematics and this area of natural science – a gap that is usually due to the language barrier and to the differences in thinking and reasoning. This book is written in a style suitable for mathematicians and adapted to their training. We have tried to remain very close to physics and to mathematics at the same time by making, in particular, a clear separation between what is assumed and what is proved.

As it is, the book may appeal as well to a broader audience, such as engineers who would like to have a different perspective on the field, relying less on physical intuition, and advanced researchers who would like an introduction to a field new to them.

The core of the book contains the fundamental parts of continuum mechanics: description of the motion of a continuous body, the fundamental law of dynamics, the Cauchy stress tensor, the constitutive laws, internal energy and the first principle of thermodynamics, shocks and the Rankine–Hugoniot relations, an introduction to fluid mechanics for inviscid and viscous Newtonian fluids, and an introduction to linear elasticity and the variational principles in linear elasticity.

Besides the core of continuum mechanics, this book also contains more or less detailed introductions to several important related fields that could be themselves the subjects of separate books: magnetohydrodynamics, combustion, geophysical fluid dynamics, vibrations, linear acoustics, and nonlinear waves and solitons in the context of the Korteweg–de Vries and the nonlinear Schrödinger equations.

This book is an extended version of an advanced undergraduate course taught several times by one of the authors (RT) and taught during the 1995–96 academic year by the other author (AM). The whole book is suitable for a one-year course at the advanced undergraduate or beginning graduate level. Parts of it are suitable for a one-semester course either on the fundamentals of continuum mechanics or on a combination of selected topics.

After briefly describing the objectives and the content of this book, let us emphasize that this book is a book on mechanics; it is not a book of mathematics, of "abstract" mechanics, or of functional analysis. There is, by choice, no functional analysis, no Sobolev or Hilbert space in this book, and the mathematical language is simple: the mathematical tools needed are those mastered by any mechanical engineer: mainly continuity, differentiation, integration, and linear algebra. The book is neither a book of "abstract" or "axiomatic" mechanics; the physical language is simple and very close to reality.

The prerequisites for all readers are calculus (differentiation, integration) and linear algebra (including some notions of tensors); however, a good knowledge of these tools is desirable.

We enjoyed writing this book, which is not at the center of our usual preoccupations, and we enjoyed the excursions we made in fields that are not ours. We hope that this book will be useful to the readers, and we welcome comments on the book.

We thank Eric Simonnet, who kindly drew all the figures in this book, and Brian Ewald and Ioana Moise for their careful reading of the manuscript and their English editing and comments; Djoko Wirosoetisno also contributed to this task. The second author (RT) benefited from very useful comments by Frédéric Abergel, Arnaud Debussche, Hervé Le Meur, and Laurent Di Menza while they assisted him in teaching parts of this book at various times, and he is grateful to Robert Kohn for providing very useful references (not fully exploited) for Chapter 16 and to Jean-Claude Saut for providing him with very useful references related to Chapters 18 and 19. We wish also to thank Jerry Bona, Philippe Ciarlet, Ciprian Foias, Tanya Leise, Jacques-Louis Lions, Morton Lowengrub, Oscar Manley, Tinsley Oden, Jay Walton, and Joseph Zyss for helpful discussions concerning the book. Of course, the continuum mechanics part of this book has been very much influenced by the French school through the teachings and the books of Henri Cabannes, Paul Germain, and Maurice Roseau and, more recently, the books of Sébastien Candel, Georges Duvaut, and Jean Salençon.

A Few Words About Notations

The notations in this book are not uniform; this is partly done on purpose and partly because we had no choice. Indeed modelers usually have to comply or at least adapt to the notations common in a given field, and thus they must be trained to some flexibility. Another reason for having nonuniform notations is that different fields are present in this volume, and it was not possible to find notations fitting "all the standards."

Another objective while deciding the notations was to choose notations that can be easily reproduced by hand writing, thus avoiding as much as possible arrows, boldfaced type, and simple and double underlining with bars or tildes; in general, in a given chapter of this book, in a given context, it is clear what a given symbol represents.

Although the notations are not rigid, there are still some repeated patterns in the notations, and we indicate hereafter notations used in several chapters:

Ω or $\mathcal{O}$, possibly with indices: domain in $\mathbb{R}^2$ or $\mathbb{R}^3$
$x = (x_1, x_2)$ or (x_1, x_2, x_3): generic point in $\mathbb{R}^2$ or $\mathbb{R}^3$. Also denoted (x, y) or (x, y, z)
$a = (a_1, a_2)$ or (a_1, a_2, a_3): initial position in Lagrangian variables
t: time
$u = (u_1, u_2)$ or (u_1, u_2, u_3), or v or w: vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Also denoted (u, v) or (u, v, w)
AB (or $\overrightarrow{AB}$ to emphasize): vector from A to B
u or U: velocity
u: displacement vector
γ: acceleration
m: mass

f, F: forces; usually f for volume forces and F for surface forces
ρ: density
g: gravity constant. Also used for equation of state for fluids
T or θ: temperature
σ: Cauchy stress tensor (in general)
n: unit outward normal on the boundary of an open set Ω or $\mathcal{O}$, $n = (n_1, n_2)$ or $n = (n_1, n_2, n_3)$

We will use also the following classical symbols and notations:

δ_{ij}: the Kronecker symbol equal to 1 if $i = j$ and to 0 if $i \neq j$

$\varphi_{,i}$ will denote the partial derivative $\partial\varphi/\partial x_i$.

The Einstein summation convention will be used: when an index (say j) is repeated in a mathematical symbol or within a product of such symbols, we add these expressions for $j = 1, 2, 3$. Hence

$$\sigma_{ij,j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad \sigma_{ij} \cdot n_j = \sum_{j=1}^{3} \sigma_{ij} n_j.$$

MATHEMATICAL MODELING IN CONTINUUM MECHANICS

Continuum mechanics is widely taught to graduate students in applied mathematics, physics, and engineering, providing the basis for further study in fluid and solid mechanics. Presentations of the subject, however, vary greatly in their level of formalism, being either engineering- and example-oriented or mathematically oversophisticated. Temam and Miranville provide a rigorous presentation of the underlying mathematics and physics of the problem, avoiding unnecessary use of function spaces. The authors then build on this base to present core topics within the general themes of fluid and solid mechanics. The brisk style allows the text to cover a wide range of topics, including viscous flows, magnetohydrodynamics, atmospheric flows, shock equations, turbulence, nonlinear solid mechanics, solitons, and the nonlinear Schrödinger equation.

This original text should be a unique resource for those studying continuum mechanics at the advanced undergraduate and beginning graduate level, whether in engineering, mathematics, physics, or the applied sciences.

Contents

PART ONE

FUNDAMENTAL CONCEPTS IN CONTINUUM MECHANICS

CHAPTER ONE

Describing the Motion of a System: Geometry and Kinematics

1.1. Deformations

The purpose of mechanics is to study and describe the motion of material systems. The language of mechanics is very similar to that of set theory in mathematics: we are interested in material bodies or systems, which are made of material points or matter particles. A material system fills some part (a subset) of the ambient space ($\mathbb{R}^3$), and the position of a material point is given by a point in $\mathbb{R}^3$; a part of a material system is called a subsystem.

We will almost exclusively consider material bodies that fill a domain (i.e., a connected open set) of the space. We will not study the mechanically important cases of thin bodies that can be modeled as a surface (plates, shells) or as a line (beams, cables). The modeling of the motion of such systems necessitates hypotheses that are very similar to the ones we will present in this book, but we will not consider these cases here.

A material system fills a domain Ω_0 in $\mathbb{R}^3$ at a given time t_0. After deformation (think, for example, of a fluid or a tennis ball), the system fills a domain Ω in $\mathbb{R}^3$. A material point, whose initial position is given by the point $a \in \Omega_0$, will be, after transformation, at the point $x \in \Omega$.

The deformation can thus be characterized by a mapping as follows (see Figure 1.1):

$$\Phi: a \in \Omega_0 \mapsto x \in \Omega.$$

Assuming that matter is conserved during the deformation, we are led to make the following natural hypothesis:

The function Φ is one-to-one from Ω_0 onto Ω.

We will further assume that the deformation Φ is a diffeomorphism of at least class C^1 from Ω_0 into Ω. In fact we assume that Φ is as smooth as needed.

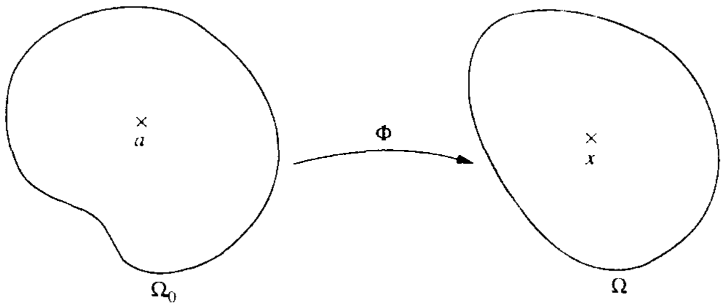

Figure 1.1: The mapping Φ.

Regularity Assumption

The regularity assumption made on Φ will actually be general; we will assume that all the functions we introduce are as regular as needed for all the mathematical operations performed to be justified (e.g., integration by parts, differentiation of an integral depending on a parameter, etc.). This hypothesis, which will be constantly assumed in the following, will only be weakened for the study of shock waves, which correspond to the appearance of discontinuity surfaces. In that case, we will assume that the map Φ is piecewise C^1.

Let grad $\Phi(a) = \nabla\Phi(a)$ be the matrix whose entries are the quantities $(\partial\Phi_i/\partial a_j)(a)$. Because Φ is a diffeomorphism, the Jacobian $\det(\nabla\Phi)$ of the transformation $a \mapsto x$ is necessarily different from zero. We will assume in the following that it is strictly positive. We will later study the role played by the linear tangent map at point a in relation to the Taylor formula

$$\Phi(a) = \Phi(a_0) + \nabla\Phi(a_0)\cdot(a - a_0) + o(|a - a_0|).$$

We will also introduce the dilation tensor to study the deformation of a "small" tetrahedron.

Displacement

Definition 1.1. *The map $u : a \mapsto x - a = \Phi(a) - a$ is called the displacement; $u(a)$ is the displacement of the particle a.*

Elementary Deformations

Our aim here is to describe some typical elementary deformations.

a) Rigid Deformations

The displacement is called rigid (in this case, we should no longer talk about deformations) when the distance between any pair of points is conserved as follows:

$$d(a, a') = d(x, x'), \qquad \forall\, a, a' \in \Omega_0,$$

where $x = \Phi(a)$, $x' = \Phi(a')$. This is equivalent to assuming that

$$\Phi \textit{ is an isometry from } \Omega_0 \textit{ onto } \Omega,$$

or, when Ω_0 is not included in an affine subspace of dimension less than or equal to 2,

$$\Phi \textit{ is an affine transformation}$$
$$\textit{(translation + rotation).}$$

In this case

$$x = L \cdot a + c, \quad c \in \mathbb{R}^3, \quad L \in \mathcal{L}_0(\mathbb{R}^3), \quad L^{-1} = L^T,$$

and

$$u(a) = (L - I)a + c,$$

where $\mathcal{L}_0(\mathbb{R}^3)$ is the space of orthogonal matrices on $\mathbb{R}^3$.

b) Linear Compression or Elongation

A typical example of elongation is given by the linear stretching of an elastic rod or of a linear spring.

Let (e_1, e_2, e_3) be the canonical basis of $\mathbb{R}^3$. The uniform elongation in the direction $e = e_1$ reads

$$x_1 = \lambda a_1, \quad x_2 = a_2, \quad x_3 = a_3,$$

with $\lambda > 1$; $0 < \lambda < 1$ would correspond to the uniform compression of a linear spring or an elastic rod. The displacement is then given by $u(a) = [(\lambda - 1)a_1, 0, 0]$ and

$$\nabla\Phi = \begin{pmatrix} \lambda - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + I.$$