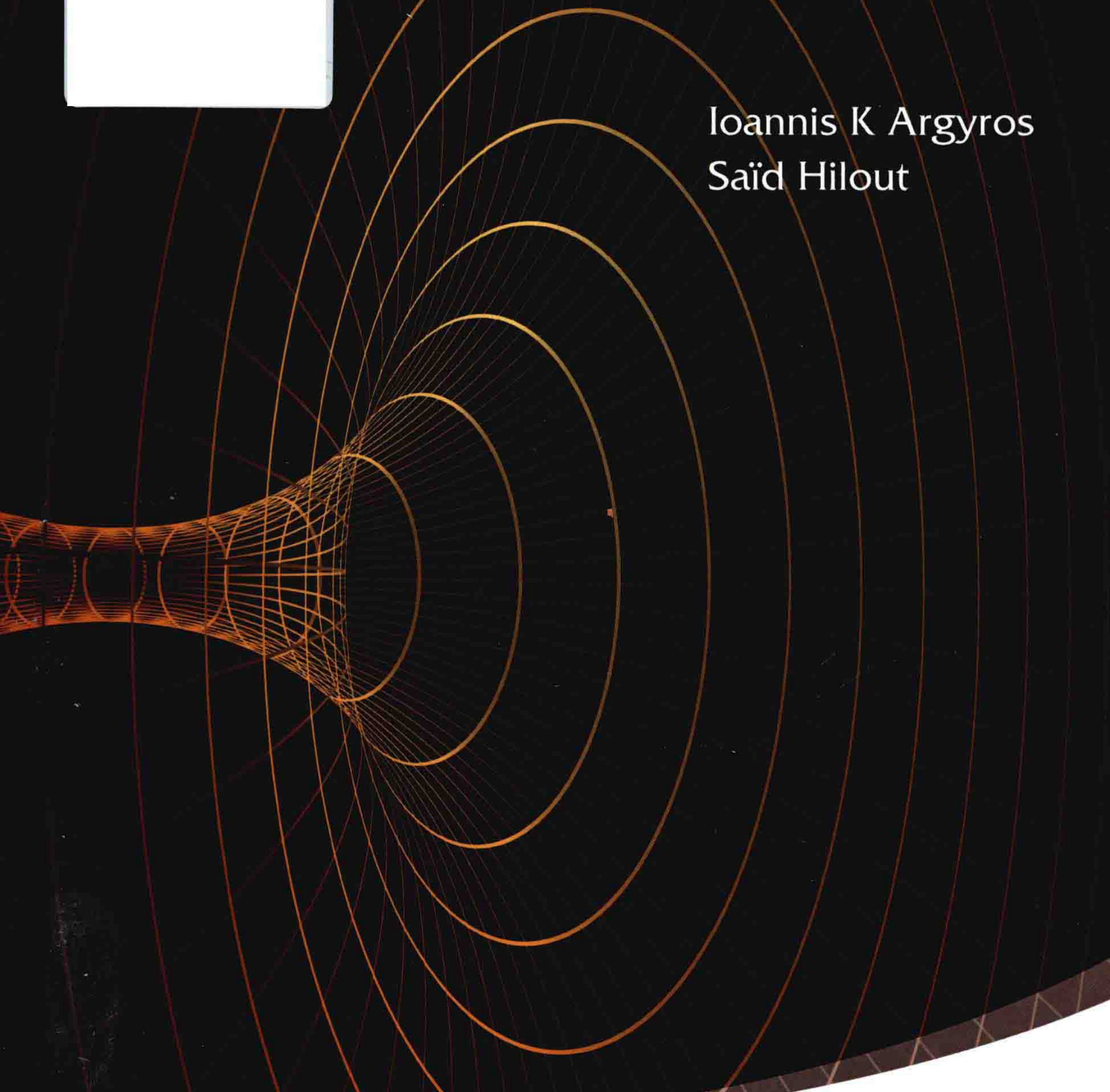
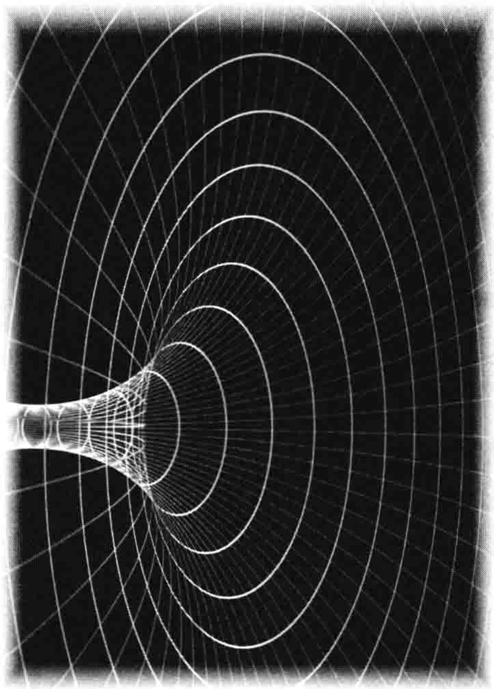
COMPUTATIONAL METHODS IN NONLINEAR ANALYSIS

Efficient Algorithms, Fixed Point Theory and Applications



Ioannis K Argyros
Saïd Hilout

World Scientific



COMPUTATIONAL METHODS IN NONLINEAR ANALYSIS

Efficient Algorithms, Fixed Point Theory and Applications

Ioannis K Argyros *(Cameron University, USA)*
Saïd Hilout *(Poitiers University, France)*

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Argyros, Ioannis K.

Computational methods in nonlinear analysis : efficient algorithms, fixed point theory and applications / by Ioannis K. Argyros (Cameron University, USA) & Saïd Hilout (Poitiers University, France).

pages cm

Includes bibliographical references and index.

ISBN 978-981-4405-82-9 (hardcover : alk. paper)

1. Nonlinear theories--Data processing. 2. Mathematics--Data processing. I. Hilout, Saïd. II. Title.

QA427.A738 2013

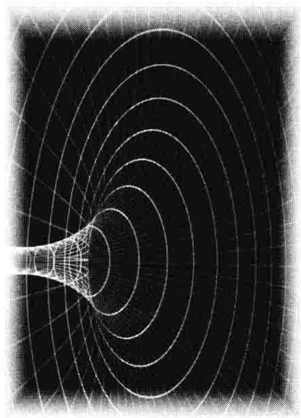
515'.355--dc23

2013005325

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

Printed in Singapore by World Scientific Printers.



COMPUTATIONAL METHODS IN NONLINEAR ANALYSIS

Efficient Algorithms, Fixed Point Theory and Applications

The first author dedicates this book to his mother Anastasia, father Konstantinos, sons Christopher, Gus and Michael and wife Diana
The second author dedicates to Aïcha, Lina, Nassim and Nouria

Preface

Many problems from applied sciences, computer graphics, engineering, optimization, economics, physics and other disciplines can be brought in the form of equations or variational inequalities using mathematical modeling. These equations or variational inequalities can be for examples: vectors (systems of linear or nonlinear algebraic equations); functions (difference, differential, integral equations); real or complex numbers (single algebraic equations with single unknowns); linear and nonlinear complementarity problems; optimality conditions for nonlinear programming. The field of computational sciences gives a lot of opportunity to researchers to solve these equations and has seen a considerable development in mathematics. The solutions of such equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The practice of numerical analysis for finding solutions is essentially connected to variants of Newton's method.

In 1669, Isaac Newton inaugurated his method through the use of numerical examples to solve equations, but did not use the current iterative expression. Later, in 1690, Raphson introduced Newton's method or the also called Newton–Raphson method. Newton's method is currently and undoubtedly the most popular one-point iterative procedure for generating a sequence approximating the solution of equation. In 1818, Fourier proved that the method converges quadratically in a neighborhood of the root, while Cauchy in 1829 and 1847 provided the multi-dimensional extension of Newton method. Kantorovich in 1948 published an important paper extending Newton's method for functional spaces.

There are usual concepts connected with iterative methods. The first concerns the evaluation of the function at each iterate to ensure that the iterates remain in the domain. In general, it is impossible to find the exact set of all initial data for which a given process is well defined and we restrict ourselves to giving conditions which guarantee that an iteration sequence is well defined for certain specific initial guesses. The secondly basic connection concerns the convergence of the sequences generated by a process and the question of whether their limit points are, in fact, solutions of the equation. The study about convergence matter of iterative methods is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of Newton's method; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There is a plethora of studies on the weakness and/or extension of the hypothesis made on the underlying operators. The economy of the entire operations and the question of how fast a given sequence will converge is also a basic connection to iterative methods. Another concept affects the best chosen method, algorithm, or software program to solve a specific type of problem and its descriptions of when a given algorithm or method succeeds or fails. Our main objective is to expand the applicability of existing iterative procedures or introduce new ones. This is being achieved by using more precise majorizing sequences than before. This approach leads to weaker sufficient convergence conditions, tighter error bounds on the distances involved and a more precise information on the location of the solutions.

This Book adopts an updated scientific approach that combines recent results in numerical methods for nonlinear equations and variational inequalities with applications in various fields of optimization, economics, control theory, engineering, linear/nonlinear differential equations, partial differential equations and physics. We present the recent results on the convergence analysis in both finite dimensional and infinite dimensional spaces. Our attention has also been paid in studying iterative procedure on manifolds since there are many numerical problems posed on manifolds that arise naturally in many contexts. The book also provides comparison between various investigations made in recent years in the field of computational sciences and connects numerical analysis with functional analysis, theory of operators and their applications. Although the book is of a theoretical nature, with optimization and weakening of existing hypotheses considerations. Each chapter contains several new theoretical results and important applications in engineering, in dynamic systems, in input-output systems, in the solution of nonlinear and linear differential equations and optimization problems. The applications appear in the form of examples or study cases or they are implied since our results improve earlier ones that have already been applied in concrete problems. Note that we have endeavored to make the main text as self-contained as possible, to prove all results in full detail. In order to make the study useful as a reference source, we have complemented each chapter with a set of remarks, comments and corollaries in which literature citations are given, other related results are discussed and various possible extensions of the results of the text are indicated. Therefore we list numerous conjectures and open problems as well as alternative models which need to be explored. The book also contains abundant and updated bibliography in the field of computational sciences.

This book is intended for researchers and practitioners in applied computational sciences, mathematical programming, engineering, optimization, mathematical economics, senior undergraduate students and graduate students. The goal is to introduce these powerful concepts and techniques at the earliest possible stage. The reader is assumed to have had basic knowledge in numerical analysis, computational linear algebra, theory of operators, functional analysis and computer programming.

Our goal in chapter 1 is to find weaker convergence criteria for Newton's method than in earlier studies. It turns out that our sufficient convergence conditions are weaker and the error bounds are tighter than in earlier studies for many interesting cases. In this chapter new results on Newton–Kantorovich theory are presented. In section 1 we use hypotheses on the first Fréchet derivative of involved operator. These new results are illustrated by several numerical examples, for which the older convergence criteria do not hold but for which our weaker convergence criteria are satisfied. In section 2 we present a two-point Newton-like method to approximate a locally unique solution of a nonlinear equation containing a non-differentiable term. In section 3 we provide new local and semi-local convergence results for Newton's method. The sufficient convergence conditions used in this section do not include the Lipschitz constant usually associated with Newton's method. In section 4 we present a local convergence analysis for Newton's method under a weak majorant condition. Our results provide under the same information a larger radius of convergence and tighter error estimates on the distances involved than before in earlier studies. Special cases and applications are also provided for validating the theoretical results. we present a local convergence analysis for the continuous analog of Newton's method. The radius of convergence is larger, the error bounds tighter and under the same or weaker hypotheses than in earlier studies. In section 5 the concept nondiscrete mathematical induction inaugurated by Potra and Ptaák is used. We extend the applicability of Newton's method for approximating a solution of nonlinear equation using weaker hypotheses. Throughout chapter 1 we present an illustrative example involving a differential equation containing a Green's kernel, a nonlinear integral equation of Chandrasekhar-type and cubic polynomial equation to show that different theorems are applicable in some situations in which the other are not applicable.

In chapter 2 we study special conditions and present some new results for convergence analysis of Newton's method in a Banach space setting. In section 1 $w^\star$-conditioned second Fréchet-derivative is explored. This way we can handle equations, where the usual Lipschitz-type conditions are non verifiable. It turns out that a similar result using ω-conditioned hypotheses can provide usable error estimates indicating only linear convergence for Newton's method. We provide in section 2 an existence and uniqueness result for nonlinear equations involving regularly smooth operators, under weaker hypotheses. Our approach extends the applicability and improves the optimality of Newton-type methods using the same information, under weaker hypotheses. Moreover, the information on the location of the solution is at least as precise as in earlier studies. We use in section 3 the special Smale's α-theory by introducing the notion of the center γ_0-condition. We expand the applicability of Newton's method. Numerical examples and applications are also provided in each section of this chapter.

Chapter 3 exhibits convergence analysis of Newton-type methods in order to approximate a zero of a mapping on special spaces as Lie groups, Hilbert space and Banach space with convergence structure. Section 1 presents Newton's method to approximate a zero of a mapping from a Lie group into its Lie algebra. Under the same computational cost and weaker convergence conditions than before. Precise information on the location of the solution is obtained in the semi-local case and larger radius of convergence is established in the local case. Note that tighter error bounds on the distances involved is obtained in both cases. In section 2 we present new semi-local convergence results for continuous modified Newton-type methods to solve nonlinear operator equations in a real Hilbert space setting. Convergence analysis is presented in section 3 to approximate a solution of equations on Banach space with convergence structure. In sections 4 to 6, for approximating zeros of a vector field on Riemannian manifolds, we present a semi-local convergence analysis of a bilinear operator free third order method, Shamanskii-type method and Traub-type method, respectively. A characterization of the convergence under Kantorovich-type conditions, error estimates and applications are also given in these three sections.

Chapter 4 presents a recent developments on Secant method. Secant method uses divided differences operator and is an attractive replacement of Newton's method studied in chapters 1 and 2. This method uses a consistent approximation operator of the Fréchet derivative and is an alternative method of Newton's method. Note also that the Secant method is of convergence order $1.6180339887\cdots$, whereas Newton's method is of order at most 2. It is a self-correcting like Newton's method and it is of high efficiency. In section 1 we extend the applicability of the method of chords in some cases. The error bounds are tighter and the information on the location of the solution at least as precise under the same information as before in earlier studies. Application and examples are also provided in this section. Section 2 exhibits the convergence analysis of Secant-type method using the concept nondiscrete mathematical induction in order to improve error bounds. A local convergence analysis for an efficient Secant-type method is given in section 3 using both the Lipschitz continuous and center-Lipshchitz continuous divided differences of order one. An estimate of the radius of the convergence ball is provided. In section 4 we use our concept of recurrent functions to provide a new semi-local convergence analysis for secant-like methods. Our convergence criteria and sufficient convergence conditions are tighter than in earlier studies. A directional Secant-type methods in Euclidean space is presented in section 5. Using weaker hypotheses and motivated by optimization considerations, a new existence convergence results are established. A numerical example to implement the method is also presented in this section. We unify in section 6 the semi-local convergence analysis of Secant-type methods under more general Lipschitz-type conditions. We present very general majorizing sequences and we extend the applicability of Secant-type methods. Our analysis includes the computation of the bounds on the limit points of the majorizing sequences involved. As in chapters 1 to 3 we present in each section of chapter 4 some remarks, examples and applications of our theoretical results.

Chapter 5 develops Gauss–Newton methods for solving nonlinear equations and convex composite optimization problems. Section 1 concerns the local convergence analysis of the iteratively regularized Gauss–Newton method for solving ill-posed problems in a Hilbert space setting. In contrast to earlier studies with Lipschitz conditions, we only use the weaker center Lipschitz conditions. Our analysis expands the applicability of this method. A semi-local convergence analysis of the Gauss–Newton method for convex composite optimization is given in section 2. We use the concept of quasi-regularity. The convergence analysis is presented under L-average Lipschitz and generalized convex majorant conditions, respectively. Our results extend the applicability of Gauss-Newton method. Section 3 presents Gauss–Newton algorithm to find local minimum of penalized nonlinear least square problems. We develop local convergence analysis from more general Lipschitz-type conditions than used in earlier studies. Our results, even in special cases, provide a larger attraction ball and tighter error estimates. In section 4 we establish a local convergence analysis of inexact Gauss–Newton like methods using more precise majorant conditions. We also provide a clearer relationship between the majorant function and the associated least squares problem.

Chapters 6, 7 and 8 contain recent results on Halley's, Chebyshev's and Broyden's methods, respectively. For Halley's method, we present semi-local and local convergence under convex majorants. Kantorovich-type and Smale-type results are considered as applications and special cases. We finish chapter 6 by a unified approach to generating majorizing sequences for multi-point iterative procedures (Traub-type method). In chapter 7 we present in section 1 semi-local convergence analysis for directional Chebyshev-type methods in finite dimensional space using recurrent relations and Newton–Kantorovich-type hypotheses. We give in section 2 of chapter 7 a semi-local convergence theorem for a new family of iterative methods obtained as a combination of the well-known Secant method and Chebyshev's method. We give a very general convergence result that allow the application of these methods to non-differentiable problems. We finish chapter 7 by finer and precise majorizing sequences for Chebyshev's method. In chapter 8 we present a new results on Broyden's method using ω-type conditions. Examples and applications are also presented in chapters 6 to 8.

The goal of chapter 9 is to give a new convergence analysis of some Newton-like methods. In section 1 we give an estimate of the convergence radius of the well-known modified Newton's method for multiple zeros, when the involved function satisfies a Hölder and center-Hölder continuity conditions. We provide in section 2 a semi-local convergence analysis for a Newton-like method under weak conditions. In particular, we only assume that the Gateaux derivative of the operator involved is hemicontinuous. Section 3 exhibits a study on the radius of convergence for some cubically convergent Newton-type method. A comparison is given between the radii of this method and Newton's method. In section 4 we present a fast two-step Newton-like method. This method unifies earlier methods such as Newton-like, Chebyshev-Secant, Chebyshev-Newton, Steffensen, Stirling's and other single or multi-step methods. We unify in section 5 the semi-local convergence analysis of Newton-like methods under more general Lipschitz-type conditions and precise majorizing sequences. Section 6 contains a new result for collection of hybrid methods combining Newton's method with frozen derivatives and a family of high order iterative schemes. We establish in section 7 the local and sem-ilocal convergence of the relaxed Newton's method, that is Newton's method with a relaxation parameter. We give a new Kantorovich-like theorem. We also obtain in this section the recurrent sequence that majorizes the one given by the method and we characterize its convergence by a result that involves the relaxation parameter. We use a new technique that allows us on the one hand to generalize and on the other hand to extend the applicability of the result given initially by Kantorovich.

In the last chapter 10 we present some new results of convergence analysis for Newton–Tikhonov methods, in order to solve ill-posed operators problems in Hilbert space. In section 1 we expand the applicability of Newton–Tikhonov method by using more precise majorizing sequence. This way we provide a semi-local convergence analysis for this method with some advantages over earlier studies. These advantages are obtained, since we use the more precise center-Lipschitz condition instead of the Lipschitz condition for the computation of the upper bounds. We also study the semi-local convergence of the simplified Newton–Tikhonov method. Section 2 develops two-step directional Newton method to solve ill-posed problems under weak conditions. An iteratively regularized projection method, which converges quadratically, has been considered in section 3 for obtaining stable approximate solution of nonlinear ill-posed operator equations. We assume in this section that only a noisy data are available. Under weaker convergence condition, a choice of the regularization parameter using an adaptive selection and a stopping rule for the iteration index using a majorizing sequence are also presented in this section.

Note that we propose in the end of each chapter exercises to apply all obtained theoretical results.

Ioannis K. Argyros and Saïd Hilout

Contents