Theoretical Global Seismology

F. A. Dahlen

and

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Preface

Five years ago, in the summer of 1993, we set out to write a slender monograph tentatively entitled *The Free Oscillations of the Earth.* With each e-mail exchange of draft chapters, our modest ambitions mushroomed; the final result is this book—*Theoretical Global Seismology*—an advanced treatise intended to be read by graduate students and researchers in geophysics and allied fields. Although the present title is more indicative of the scope than the original, the contents nevertheless reflect the book's origins. We devote our attention almost exclusively to the forward problem of computing synthetic seismograms upon a realistic three-dimensional model of the Earth, with a strong emphasis on the normal-mode summation method. Free oscillations have many close associations with surface waves, and we consider them in some detail as well; we give shorter shrift to body waves, and do not discuss seismic instrumentation, data analysis procedures, or geophysical inverse theory at all.

The introductory chapter recounts the history of free-oscillation and surface-wave research, beginning with the earliest theoretical investigation of the oscillations of an elastic sphere in the 1820's, through the first observation of the gravest terrestrial oscillations following the great Chile earthquake of 1960, and concluding with the initial determinations of global upper-mantle heterogeneity using digitally recorded seismograms in the 1980's. The remainder of the text—like ancient Gaul—is divided into three parts. In Part I—*Foundations*—we derive the linearized equations of motion governing both an elastic and anelastic Earth subject to a non-hydrostatic state of initial stress, and show how to express the elastic-gravitational response to an arbitrary earthquake source as a sum of free oscillations or normal modes. We conclude with a discussion of the Rayleigh-Ritz method, which yields a truncated matrix formulation that is identical to the classical theory of the small oscillations of a system with a finite number of degrees of freedom, generalized to account for rotation and anelasticity. In Part II—*The Spherical Earth*—we restrict attention to the case of an Earth model that is non-rotating and spherically symmetric; the toroidal and spheroidal eigenfrequencies and eigenfunctions of such a model can be found essentially exactly by numerical integration of the governing radial differential equations. We show how to calculate synthetic seismograms on a spherical Earth by means of normal-mode summation, and discuss the propagation of Love and Rayleigh surface waves as well as mode-ray duality. These results form the basis for the more general considerations in Part III—*The Aspherical Earth*—where we use perturbation theory to treat the splitting and coupling of the normal-mode multiplets produced by the Earth's rotation, ellipticity and other departures from spherical symmetry, and JWKB theory to describe the propagation of both body waves and surface waves upon a laterally heterogeneous Earth.

The three parts are arranged in order of decreasing "shelf life". The fundamental equations and results obtained in Part I are applicable to a very general Earth model, and should provide the basis for discussions of the elastic-gravitational deformation of the Earth into the foreseeable future. The results pertaining to a spherical Earth in Part II are likewise well established; only relatively minor numerical details are likely to change as the spherically averaged structure of the Earth continues to be refined. The approximate methods of dealing with the Earth's lateral heterogeneity which we discuss in Part III are not as well developed; three-dimensional global tomography is an extremely active research field at the present time, and improvements in the procedures and results described here seem likely in the future. In addition to the fifteen chapters in Parts I through III, there are four mathematical appendixes devoted to vectors and tensors, ordinary and generalized spherical harmonics, and the matrix machinery needed to calculate coupled-mode synthetic seismograms on a rotating, anelastic, laterally heterogeneous Earth.

Variational principles appear in a number of guises, and provide a unifying thread which serves to knit the various chapters together. We enunciate Hamilton's principle for a general elastic Earth model in Chapter 3, and discuss its frequency-domain analogue, Rayleigh's principle, on both a non-rotating and rotating Earth in Chapter 4. We extend Rayleigh's principle to an anelastic Earth in Chapter 6, deduce the equivalent elastic and anelastic matrix principles in Chapter 7, and utilize the orthonormality of the surface spherical harmonics to obtain a purely radial variational principle on a spherically symmetric Earth in Chapter 8. The one-dimensional and three-dimensional versions of Rayleigh's principle provide the basis for the spherical and aspherical perturbation analyses in Chapters 9 and 13. Finally, we develop ray theory for body waves and JWKB theory for surface waves on a smooth laterally heterogeneous Earth using an associated slow variational principle in Chapters 15 and 16.

The subject matter of this book may be described as mathematical in the sense that there is a high proportion of equations to words; however, all of the theoretical considerations are purely formal, with no attempt at rigor whatsoever. We are not finicky about the continuity and differentiability of displacement, strain and stress fields, or the open or closed nature of regions within the Earth, except where it matters to get the physics right. The only mathematical property of the elastic-gravitational operator governing the Earth's free oscillations which is considered to be physically significant is whether or not it is Hermitian. We blithely assume that the normal modes of an elastic Earth model are complete, ignore the presence of a branch cut in developing a mode-sum representation of the response of an anelastic Earth, manipulate infinite matrices without regard for convergence, and seldom worry about the precise nature of the equality in spherical-harmonic and other infinite orthonormal eigenfunction expansions.

Sections denoted by a star* contain more esoteric material which may be omitted upon a first reading. Many of the starred sections deal with the theoretical complications introduced by the Earth's rotation; for example, in analyzing the influence of anelasticity upon the free oscillations, it is necessary to introduce the dual *eigenfunctions* $\bar{\mathbf{s}}$ associated with the "anti-Earth" having the opposite sense of rotation, as well as the ordinary eigenfunctions $\mathbf{s}$ of the Earth itself. A few unstarred sections make use of these dual eigenfunctions in the interest of brevity and maximum generality; uninterested readers may simply eliminate the overbars, since in the absence of rotation the eigenfunctions and their duals coincide: $\bar{\mathbf{s}} = \mathbf{s}$.

We are deeply indebted to many colleagues for their generous support and assistance during the preparation of this book. First and foremost, we would like to thank Freeman Gilbert for his barrage of encouraging e-mails, filled with valuable commentary upon a variety of topics—ranging from the inherent positivity of the group speed to the application of ray theory to stealth-aircraft detection. We also wish to express our sincere gratitude to Guust Nolet, whose detailed and constructive criticism, particularly of the appendixes, was extremely helpful. An early, incomplete draft was reviewed by Brian Kennett, Guy Masters and Barbara Romanowicz; they suggested a number of improvements which have been incorporated in the final version. Several people graciously complied with our request to read a particular chapter devoted to their field of expertise; we especially wish to thank Henk Marquering and Roel Snieder for their remarks regarding Chapter 11, Li Zhao for his careful review of Chapter 12, and Colin Thomson for his advice on Chapter 15. Appendices B and C on ordinary and generalized spherical harmonics are based in part upon lecture notes by George Backus and John Woodhouse. Further suggestions for improvements and additions were provided by Chris Chapman, Adam Dziewonski, Andy Jackson, Paul Richards, and Philippe Lognonné. Our foray into the early German surface-wave literature was aided by Thomas Meier. Finally, we would like to acknowledge our indebtedness to Miaki Ishii, whose thorough review of the entire manuscript is very much appreciated.

We are grateful to many individuals for helping us to assemble the more than 225 illustrations. The theoretical spectra and seismograms and eigenfunction and Fréchet kernel plots were almost all produced by students in Jeroen Tromp's global seismology courses at Harvard and MIT. We thank John He, Yu Gu, Rishi Jha, Hrafnkell Kárason, Erik Larson, Xian-Feng Liu, Jeff McGuire, Meredith Nettles, Frederik Simons and Mark Taylor for their help in this endeavor. A number of colleagues, including Göran Ekström, Guy Masters, Joe Resovsky, Mike Ritzwoller, Barbara Romanowicz, Genevieve Roult, Peter Shearer, Zheng Wang, Shingo Watada, Ruedi Widmer and Li Zhao, provided us with additional figures; we sincerely thank them all. Most of the cartoons were ably drafted from our slapdash sketches by Dearbhla McHenry and Leslie Hsu; the indispensable Leslie also organized, re-sized, touched-up and unified all of the figures for encapsulation into the final camera-ready copy.

The labor of composing, formatting and typesetting this behemoth of more than 1000 pages and 3800 numbered equations was ameliorated by LaTeX, BibTeX and *MakeIndex*; we benefitted from the expertise of Bob Fischer and Erik Larson. Meredith Nettles and Yu Gu indulged our paranoia by religiously backing up all of the chapter and figure files. It has been a pleasure working with the capable staff at Princeton University Press, particularly Jack Repcheck, who has guided this book to publication from the outset, and Jennifer Slater, who did a splendid job of copy editing.

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Finally, we would like to thank Elisabeth, Tracey and Alex for patiently putting up with our seismological gibberish and preoccupation with this project during the past five years. The fact of the matter is that we should be grateful that they put up with us at all.

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Theoretical Global Seismology

Chapter 1

Historical Introduction

After every major earthquake, the Earth rings like a large bell for several days. These free oscillations of the Earth are routinely detected at modern broad-band seismographic stations, which are now distributed globally. The eigenfrequencies and decay rates of the vibrations can be measured and used to constrain the radial and lateral distribution of density, seismic wave speed and anelastic attenuation within the interior. The observed amplitudes and phases can likewise be used to infer the origin times, hypocentral locations, seismic moments, and fault geometries of the earthquakes responsible for the excitation. The analysis of the free oscillations of the Earth and the allied normal-mode methods employed in the determination of the Earth's internal structure and the source mechanisms of earthquakes—the topics considered in this book—constitute one of the cornerstones of quantitative seismology. Excellent reviews which summarize the state of progress at two pivotal points in the development of the field are provided by Stoneley (1961), Lapwood & Usami (1981) and Buland (1981). This introduction contains our own brief historical survey of research on terrestrial free oscillations and the associated propagating surface waves, focusing upon the theoretical and observational advances made prior to 1985. More recent developments are described—with little attention to their historical context—in subsequent chapters of the book.

1.1 Early Theoretical Studies

The theoretical analysis of the Earth's normal modes was initiated over one and one-half centuries ago by the French mathematician Poisson. In a remarkable memoir presented to the Paris Academy of Sciences in August 1828, he developed a general theory of deformation for solid materials, based upon "la considération des actions mutuelles de leurs molécules", and applied it to a large number of special elastostatic and elastodynamic problems, including the determination of the frequencies of the purely radial oscillations of a homogeneous, non-gravitating sphere (Poisson 1829). These investigations, together with the work of his contemporaries Navier and Cauchy, laid the foundations for the modern theory of linear elasticity. The equations of equilibrium and vibration derived by Poisson are now recognized to be incomplete, inasmuch as they characterize the elastic response of an isotropic solid in terms of a single elastic parameter rather than two; the radial-mode eigenfrequencies and eigenfunctions he obtained are, however, correct in the special case that we now refer to as a Poisson solid, which has $\kappa = \frac{5}{3}\mu$, where κ is the incompressibility and μ is the rigidity. Not being a physicist or natural philosopher, Poisson did not seek to estimate or calculate the numerical free periods of radial vibration of the Earth or any man-made spherical objects, but rather expressed his final results in terms of dimensionless ratios.

The first numerical estimate of a vibrational eigenfrequency of the Earth was made by Lord Kelvin in 1863. The prevailing opinion of most geologists and geophysicists at the time was that the Earth was completely molten, except for a thin crust of solid rock. Supporting evidence for this conclusion included the good agreement of the observed ellipticity of figure with the hydrostatic theory of Clairaut, the rapid increase of temperature with depth in mines, and the eruption of lava from active volcanoes. Seeking to challenge this view, Kelvin calculated the fundamental degree-two spheroidal-mode eigenfrequency of the Earth using two different assumptions (Thomson 1863a). For a self-gravitating fluid Earth he found the period of this mode—now designated ${}_0S_2$—to be 94 minutes, whereas for a solid Earth having the same rigidity as steel he asserted that the period would be approximately 69 minutes. The first value was obtained by means of an exact dynamical analysis for a homogeneous, incompressible ($\kappa = \infty$) fluid ($\mu = 0$) sphere (Thomson 1863b), whereas the second was estimated on the basis of the time required for a shear wave to transit the diameter, using a laboratory value for the rigidity of steel obtained from his brother James in Glasgow. Lacking a means to measure the terrestrial eigenfrequencies, Kelvin devised an ingenious procedure for determining the mean rigidity of the Earth based upon the height of the fortnightly and monthly tides. He noted that the gravitational attraction of the Moon and Sun must raise bodily tides within the solid Earth as well as the tides within the oceans familiar to all seafarers, and pointed out that the observed oceanic tides, which are measured with respect to the deformed seafloor, should be nearly zero on a molten Earth. He determined the elastic-gravitational response of a homogeneous, incompressible solid sphere to an applied tidal potential, and showed that the oceanic tides on an elastic Earth should be reduced relative to their equilibrium value on a rigid Earth by an amount $\eta = (19\mu/2\rho g a)(1 + 19\mu/2\rho g a)^{-1}$, where ρ is the density, a is the radius, and g is the surficial acceleration of gravity. Since Kelvin's analysis was quasi-static, this elastic-Earth reduction factor could not be applied directly to the dominant semi-diurnal and diurnal tides; however, he argued that it should be applicable to the fortnightly and monthly tides, since they are largely devoid of ocean-basin resonance. The available fortnightly and monthly observations were insufficiently accurate for his purpose; accordingly, he persuaded the British Association to establish a Tidal Committee charged with "the evaluation of the long-period tides for the purpose of answering the question of the Earth's rigidity". The harmonic analysis of 66 years of tidal observations from fourteen British, French and Indian ports was undertaken by George Darwin, who published his results in the second edition of the *Treatise on Natural Philosophy* (Thomson & Tait 1883). Averaging the results from all ports and both tides, Darwin found that $\eta = 0.676 \pm 0.076$, indicating that the tidal-effective rigidity of the Earth is indeed "about equal to that of steel". This celebrated conclusion corroborated Kelvin's 69-minute estimate of the period of the ${}_0S_2$ mode, grounding it upon a measured physical property of the Earth.

An early theoretical investigation of the toroidal modes of a homogeneous sphere was undertaken by Jaerisch (1880); however, the first comprehensive treatment of the free oscillations of a non-gravitating sphere is the classic analysis of Lamb (1882). He distinguished clearly between the spheroidal oscillations, which he called "vibrations of the first class", and the toroidal oscillations, which he called "vibrations of the second class", and concluded that the period of the ${}_0S_2$ mode for a steel sphere the size of the Earth should be 65 minutes in the case $\kappa = \infty$ and 66 minutes in the case $\kappa = \frac{5}{3}\mu$. The good agreement with Kelvin's order-of-magnitude estimate is to some extent coincidental, since Lamb used an improved, slightly higher value for the rigidity μ of steel; the insensitivity to the value of the incompressibility κ is a consequence of the fact that the ellipsoidal deformation is dominated by shear. Lamb conducted his analysis in terms of three-dimensional Cartesian coordinates; however, it was subsequently shown by Chree (1889) that the same results could be obtained much more economically using spherical polar coordinates. Such a spherical-harmonic representation of the elastic-gravitational deformation of the Earth has been employed in the majority of theoretical analyses ever since.

The proximity of the two rigorously derived periods—94 minutes for a fluid sphere whose only restoring force is the mutual gravitation of its parts and 65 minutes for a Poisson-solid sphere devoid of gravitational