GENERALIZED VECTOR AND DYADIC ANALYSIS

APPLIED MATHEMATICS IN FIELD THEORY

CHEN-TO TAI

IEEE PRESS

Generalized Vector and Dyadic Analysis

Applied Mathematics in Field Theory

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IEEE Antennas and Propagation Society, *Sponsor*

IEEE
PRESS

The Institute of Electrical and Electronics Engineers, Inc., New York

IEEE PRESS
445 Hoes Lane, PO Box 1331
Piscataway, NJ 08855-1331

ISBN 0-87942-288-2
IEEE Order Number: PC0283-2

Printed in the United States of America

10 9 8 7 6 5 4 3 2

Library of Congress Cataloging-in-Publication Data

Tai, Chen-To (date)
Generalized vector and dyadic analysis / Chen-To Tai.
p. cm.
"IEEE order number: PC0283-2"—T.p. verso.
Includes bibliographical references and index.
ISBN 0-87942-288-2
1. Vector analysis. I. Title. II. Title: Dyadic analysis.
QA433.T3 1992
515'.63—dc20 91-23537

In Memory
of
Professor Dr. Yeh Chi-Sun
(1898–1977)

Preface

Mathematics is a language.

The whole is simpler than its parts.

Anyone having these desires will make these researches.

—J. Willard Gibbs

This monograph is mainly based on the author's recent work on vector analysis and dyadic analysis. The book is divided into two main topics: Chapters 1–6 cover vector analysis, while Chapter 7 is exclusively devoted to dyadic analysis. On the subject of vector analysis, a new symbolic method with the aid of a symbolic vector is the main feature of the presentation. By means of this method, the principal topics in vector analysis can be developed in a systematic way. All vector identities can be derived by an algebraic manipulation of expressions with two partial symbolic vectors without actually performing any differentiation. Integral theorems are formulated under one roof with the aid of a generalized Gauss theorem. Vector analysis on a surface is treated in a similar manner. Some basic differential functions on a surface are defined; they are different from the surface functions previously defined by Weatherburn, although the two sets are intimately related. Their relations are discussed in great detail. The advantage of adopting the surface functions advocated in this work is the simplicity of formulating the surface integral theorems based on these newly defined functions.

The scope of topics covered in this book on vector analysis is comparable to those found in the books by Wilson [21], Gans [4], and Phillips [11]. However, the topics on curvilinear orthogonal systems have been treated in great detail. One important feature of this work is the unified treatment of many theorems and formulas of similar nature, which includes the invariance principle of the differential operators for the gradient, the divergence, and the curl, and the relations between various integral theorems and transport theorems. Some quite useful topics are found in this book, which include the derivation of several identities involving the derivatives of unit vectors, and the relations between the unit vectors of various coordinate systems based on a method of gradient.

Tensor analysis is outside the scope of this book. There are many excellent books treating this subject. Since dyadic analysis is now used quite frequently in engineering sciences, a chapter on this subject, which is closely related to tensor analysis in a three-dimensional Euclidean space, may be timely.

As a whole, it is hoped that this book may be useful to instructors and students in engineering and physical sciences who wish to teach and to learn vector analysis in a systematic manner based on a new method with a clear picture of the constituent structure of this mature science not critically studied in the past few decades.

Acknowledgment

Without the encouragement which I received from my wife and family, and the loving innocent interference from my grandchildren, this work would never have been completed. I would like to express my gratitude to President Dr. Qian Wei-Chang for his kindness in inviting me as a Visiting Professor at The Shanghai University of Technology in the Fall of 1988 when this work was started. Most of the writing was done when I was a Visiting Professor at The Chung Cheng Institute of Technology, Taiwan, in the Spring of 1990. I am indebted to President Dr. Chen Chwan-Haw, Prof. Bor Sheau-Shong, and Prof. Kuei Ching-Ping for the invitation.

The assistance of Prof. Nenghang Fang of The Nanjing Institute of Electronic Technology, China, currently a Visiting Scholar at The University of Michigan, has been most valuable. His discussion with me about the Russian work on vector analysis was instrumental in stimulating my interest to formulate the symbolic vector method introduced in this book. Without his participation in the early stage of this work, the endeavor could not have begun. He has kindly checked all the formulas and made numerous suggestions. I am grateful to many colleagues for useful information and valuable comments. They include: Prof. J. Van Bladel of The University of Gent, Prof. Jed Z. Buchwald of The University of Toronto, Prof. W. Jack Cunningham of Yale University, Prof. Walter R. Debler and Prof. James F. Driscoll of The University of Michigan, Prof. John D. Kraus and Prof. H. C. Ko of The Ohio State University, and Prof. C. Truesdell of The Johns Hopkins University. My dear old friend Prof. David K. Cheng of Syracuse University kindly edited the manuscript and suggested the title of the book. The teachings of Prof. Chih-Kung Jen of The Johns Hopkins Applied

Physics Laboratory, formerly of Tsing Hua University, and Prof. Ronold W. P. King of Harvard University remain the guiding lights in my search for knowledge. Without the help of Ms. Bonnie Kidd, Dr. Jian-Ming Jin, and Dr. Leland Pierce, the preparation of this manuscript would not have been so professional and successful.

I wish to thank Prof. Fawwaz T. Ulaby, Director of the Radiation Laboratory, for providing me with technical assistance. The speedy production of this book is due to the efficient management of Mr. Dudley Kay, Executive Editor, and the valuable technical supervision of Ms. Anne Reifsnyder, Associate Editor, of the IEEE Press. Some major changes have been made in the original manuscript as a result of many valuable suggestions from the reviewers. I am most grateful to these reviewers.

CHEN-TO TAI

Ann Arbor, MI

Publisher's Acknowledgment

The IEEE Press would like to thank Dr. Wilson Pearson of Clemson University, Antennas and Propagation Society Liaison to the Press, for his support of the work. We would also like to thank Robert D. Nevels of Texas A & M University, Glenn S. Smith of Georgia Institute of Technology, and N. G. Alexopoulos of the University of California, Los Angeles for their constructive comments as reviewers of the manuscript.

Contents

CHAPTER 1

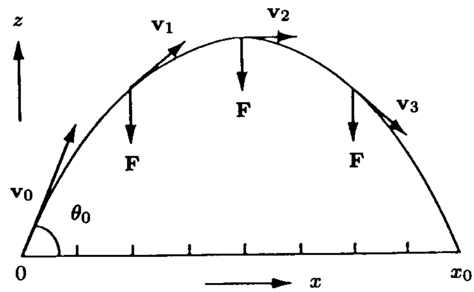
Vector Algebra

1-1 REPRESENTATIONS OF VECTOR FUNCTIONS

A vector function has both magnitude and direction. The vector functions which we encounter in many physical problems are, in general, functions of space and time. In the first five chapters, we discuss only their characteristics as functions of spatial variables. Functions of space and time are covered in Chapter 6 dealing with a moving surface or a moving contour.

A vector function is denoted by $\mathbf{F}$. Geometrically, it is represented by a line with an arrow in a three-dimensional space. The length of the line corresponds to its magnitude, and the direction of the line represents the direction of the vector function. The convenience of using vectors to represent physical quantities is illustrated by a simple example shown in Fig. 1-1 which describes the motion of a mass particle in a frictionless air (vacuum) against a constant gravitational force. The particle is thrown into the space with an initial velocity $\mathbf{v}_0$, making an angle θ_0 with respect to the horizon. During its flight, the velocity function of the particle changes both its magnitude and direction, as shown by $\mathbf{v}_1, \mathbf{v}_2$, etc., at subsequent locations. The gravitational force which acts on the particle is assumed to be constant, and it is represented by $\mathbf{F}$ in the figure. A constant vector function means that both the magnitude and the direction of the function are constant, being independent of the spatial variables, x and z in this case.

The rule of the addition of two vectors $\mathbf{a}$ and $\mathbf{b}$ is shown geometrically by Fig. 1-2 (a), (b), or (c). Algebraically, it is written in the same form as the addition of two numbers of two scalar functions, i.e.,



$$z = x\left(1 - \frac{x}{x_0}\right)\tan\theta_0$$

$$x_0 = \frac{1}{g}v_0^2 \sin 2\theta_0$$

$g =$ gravitational constant

Fig. 1-1 Trajectory of a mass particle in a gravitational field showing the velocity $\mathbf{v}$ and the constant force vector $\mathbf{F}$ at different locations.

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{1.1}$$

The subtraction of vector **b** from vector **a** is written in the form

$$\mathbf{d} = \mathbf{a} - \mathbf{b}. \tag{1.2}$$

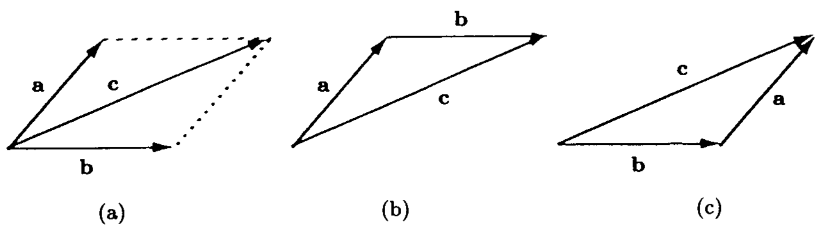


Fig. 1-2 Addition of vectors, $\mathbf{a} + \mathbf{b} = \mathbf{c}$.

Now, $-\mathbf{b}$ is a vector which has the same magnitude as **b**, but of opposite direction; then (1.2) can be considered as the addition of **a** and $(-\mathbf{b})$. Geometrically, the meaning of (1.2) is shown in Fig. 1-3. The sum and the difference of two vectors obey the associate rule, i.e.,

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \tag{1.3}$$

and

$$\mathbf{a} - \mathbf{b} = -\mathbf{b} + \mathbf{a}. \tag{1.4}$$

They can be generalized to any number of vectors.

The rule of the addition of vectors suggests that any vector can be considered as being made of basic components associated with a proper coordinate

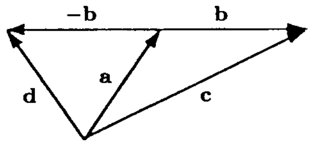


Fig. 1-3 Subtraction of vectors, $\mathbf{a} - \mathbf{b} = \mathbf{d}$.

system. The most convenient system to use is the Cartesian system or the rectangular coordinate system. The spatial variables in this system are commonly denoted by x, y, z. A vector which has a magnitude equal to unity and pointed in the positive x direction is called a unit vector in the x direction and is denoted by $\mathbf{u}_x$. Similarly, we have $\mathbf{u}_y, \mathbf{u}_z$. In such a system, a vector function $\mathbf{F}$ which, in general, is a function of position, can be written in the form

$$\mathbf{F} = F_x\mathbf{u}_x + F_y\mathbf{u}_y + F_z\mathbf{u}_z. \tag{1.5}$$

The three scalar functions F_x, F_y, F_z are called the components of $\mathbf{F}$ in the direction of $\mathbf{u}_x, \mathbf{u}_y$, and $\mathbf{u}_z$, respectively, while $F_x\mathbf{u}_x, F_y\mathbf{u}_y$, and $F_z\mathbf{u}_z$ are called the vector components of $\mathbf{F}$. The geometrical representation of $\mathbf{F}$ is shown in Fig. 1-4. It is seen that F_x, F_y, and F_z can be either positive or negative. In Fig. 1-4, F_x and F_z are positive, but F_y is negative.

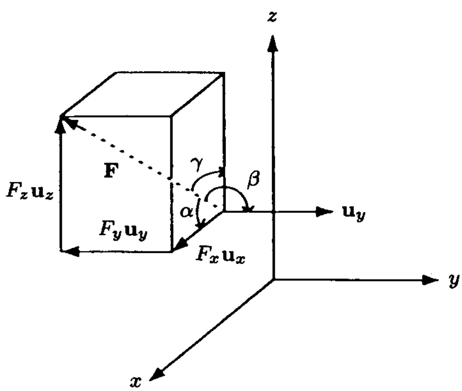


Fig. 1-4 Components of a vector Cartesian system.

In addition to the representation by (1.5), it is sometimes desirable to express $\mathbf{F}$ in terms of its magnitude, denoted by $|\mathbf{F}|$, and its directional cosines, i.e.,

$$\mathbf{F} = |\mathbf{F}|\left(\cos\alpha\mathbf{u}_x + \cos\beta\mathbf{u}_y + \cos\gamma\mathbf{u}_z\right). \tag{1.6}$$

zα, β, and γ are the angles which $\mathbf{F}$ makes, respectively, with $\mathbf{u}_x, \mathbf{u}_y$, and $\mathbf{u}_z$, as shown in Fig. 1-4. It is obvious from the geometry of that figure that

$$|\mathbf{F}| = \left(F_x^2 + F_y^2 + F_z^2\right)^{\frac{1}{2}} \tag{1.7}$$

and

$$\cos\alpha = \frac{F_x}{|\mathbf{F}|}, \cos\beta = \frac{F_y}{|\mathbf{F}|}, \cos\gamma = \frac{F_z}{|\mathbf{F}|}. \tag{1.8}$$

Furthermore, we have the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \tag{1.9}$$

In view of (1.9), only two of the directional cosine angles are independent. From the above discussion, we observe that, in general, we need three parameters to specify a vector function. The three parameters could be F_x, F_y, and F_z or $|\mathbf{F}|$ and two of the directional cosine angles. Representations such as (1.5) and (1.6) can be extended to other orthogonal coordinate systems which will be discussed in a later chapter.

1-2 PRODUCTS AND IDENTITIES

The scalar product of two vectors **a** and **b** is denoted by $\mathbf{a} \cdot \mathbf{b}$ and it is defined by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta \tag{1.10}$$

where θ is the angle between **a** and **b**, as shown in Fig. 1-5. Because of the notation used for such a product, sometimes it is called the dot product. By applying (1.10) to three orthogonal unit vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, one finds

$$\mathbf{u}_i \cdot \mathbf{u}_j = \begin{Bmatrix} 1, & i = j \\ 0, & i \neq j \end{Bmatrix}, \qquad i, j = 1, 2, 3. \tag{1.11}$$

The value of $\mathbf{a} \cdot \mathbf{b}$ can also be expressed in terms of the components of **a** and **b** in any orthogonal system. Let the system under consideration be the Cartesian system, and let $\mathbf{c} = \mathbf{a} - \mathbf{b}$; then

$$|\mathbf{c}|^2 = |\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta.$$

Hence,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}|\,|\mathbf{b}|\cos\theta = \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2}{2} \\ &= \frac{a_x^2 + a_y^2 + a_z^2 + b_x^2 + b_y^2 + b_z^2 - (a_x - b_x)^2 - (a_y - b_y)^2 - (a_z - b_z)^2}{2} \\ &= a_x b_x + a_y b_y + a_z b_z. \end{aligned} \tag{1.12}$$

By equating (1.10) and (1.12), one finds

$$\begin{aligned} \cos\theta &= \frac{1}{|\mathbf{a}|\,|\mathbf{b}|}(a_x b_x + a_y b_y + a_z b_z) \\ &= \cos\alpha_a \cos\alpha_b + \cos\beta_a \cos\beta_b + \cos\gamma_a \cos\gamma_b, \end{aligned} \tag{1.13}$$

a relationship well known in analytical geometry. Equation (1.12) can be used to prove the validity of the distributive law for the scalar products, namely,

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}. \tag{1.14}$$

According to (1.12), we have

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} &= (a_x+b_x)\,c_x+(a_y+b_y)\,c_y+(a_z+b_z)\,c_z\\ &= (a_xc_x+a_yc_y+a_zc_z)+(b_xc_x+b_yc_y+b_zc_z)\\ &= \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}.\end{aligned}$$

Once we have proved the distributive law for the scalar product, (1.12) can be verified by taking the sum of the scalar products of the individual terms of **a** and **b**.

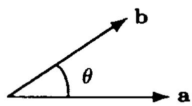


Fig. 1-5 Scalar product of two vectors, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$.

The vector product of two vector functions **a** and **b**, denoted by $\mathbf{a}\times\mathbf{b}$, is defined by

$$\mathbf{a}\times\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\sin\theta\mathbf{u}_c \tag{1.15}$$

where θ denotes the angle between **a** and **b**, measured from **a** to **b**; $\mathbf{u}_c$ denotes a unit vector perpendicular to both **a** and **b** and is pointed to the advancing direction of a right-hand screw when we turn from **a** to **b**. Figure 1-6 shows the relative position of $\mathbf{u}_c$ with respect to **a** and **b**. Because of the notation used for the vector product, it is sometimes called the cross product, in contrast to the dot product or the scalar product. For three orthogonal unit vectors in a right-hand system, we have $\mathbf{u}_1\times\mathbf{u}_2=\mathbf{u}_3, \mathbf{u}_2\times\mathbf{u}_3=\mathbf{u}_1$, and $\mathbf{u}_3\times\mathbf{u}_1=\mathbf{u}_2$. It is obvious that $\mathbf{u}_i\times\mathbf{u}_i=0, i=1,2,3$. From the definition of the vector product defined by (1.15), one finds

$$\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}. \tag{1.16}$$

The value of $\mathbf{a}\times\mathbf{b}$ as described by (1.15) can also be expressed in terms of the components of **a** and **b** in a Cartesian coordinate system. If we let $\mathbf{a}\times\mathbf{b}=\mathbf{v}=v_x\mathbf{u}_x+v_y\mathbf{u}_y+v_z\mathbf{u}_z$, which is perpendicular to both **a** and **b**, then

$$\mathbf{a}\cdot\mathbf{v} = a_xv_x+a_yv_y+a_zv_z=0 \tag{1.17}$$

$$\mathbf{b}\cdot\mathbf{v} = b_xv_x+b_yv_y+b_zv_z=0. \tag{1.18}$$

Solving for v_x/v_z and v_y/v_z, from (1.17) and (1.18) we obtain

$$\frac{v_x}{v_z}=\frac{a_yb_z-a_zb_y}{a_xb_y-a_yb_x},\ \frac{v_y}{v_z}=\frac{a_zb_x-a_xb_z}{a_xb_y-a_yb_x}.$$

Thus,

$$\frac{v_x}{a_yb_z-a_zb_y}=\frac{v_y}{a_zb_x-a_xb_z}=\frac{v_z}{a_xb_y-a_yb_x}.$$

Let the common ratio of these quantities be denoted by c, which can be determined by considering the case with $\mathbf{a} = \mathbf{u}_x, \mathbf{b} = \mathbf{u}_y$; then $\mathbf{v} = \mathbf{a} \times \mathbf{b} = \mathbf{u}_z$; hence, from the last ratio, we find $c = 1$ because $v_z = 1$ and $a_x = b_y = 1$ while $a_y = b_x = 0$. The three components of $\mathbf{v}$, therefore, are given by

$$\left.\begin{aligned} v_x &= a_y b_z - a_z b_y \\ v_y &= a_z b_x - a_x b_z \\ v_z &= a_x b_y - a_y b_x \end{aligned}\right\} \tag{1.19}$$

which can be assembled in a determinant form as

$$\mathbf{v} = \begin{vmatrix} \mathbf{u}_x & \mathbf{u}_y & \mathbf{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}. \tag{1.20}$$

We can use (1.20) to prove the distributive law of vector products, i.e.,

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}. \tag{1.21}$$

To prove (1.21), we find that the x component of $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ according to (1.20) is equal to

$$\begin{aligned} &(a_y + b_y)\, c_z \;-\; (a_z + b_z)\, c_y \\ &= (a_y c_z - a_z c_y) \;+\; (b_y c_z - b_z c_y)\,. \end{aligned} \tag{1.22}$$

The last two terms in (1.22) denote, respectively, the x component of $\mathbf{a} \times \mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$. The equality of the y and z components of (1.21) can be proved in a similar manner.

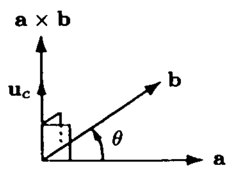


Fig. 1-6 Vector product of two vectors, $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \mathbf{u_c}$; $\mathbf{u_c} \perp \mathbf{a}$, $\mathbf{u_c} \perp \mathbf{b}$.

In addition to the scalar product and the vector product introduced before, there are two identities involving the triple products that are very useful in vector analysis. They are

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \tag{1.23}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}. \tag{1.24}$$

Identities described by (1.23) can be proved by writing $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ in a determinant form:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$