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J-P Aubin & R B Vinter (Editors)

Convex analysis and optimization

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Convex analysis and optimization

Preface

The following papers by J.-P. Aubin, J.-P. Crouzeix, I. Ekeland, J.-B. Hiriart-Urruty, J.E. Jayne and C.A. Rogers, R.B. Vinter and L.C. Young are expanded versions of presentations at the Colloquium

CONVEX ANALYSIS AND OPTIMIZATION

held in honour of Alexander D. Ioffe at Imperial College, London, on 28/29 February 1980.

The papers by J.-P. Aubin and I. Ekeland are of an expository nature and concern non-smooth analysis. Considerable interest has been shown recently in finding an appropriate definition of a generalized derivative of a locally Lipschitzian function taking values in a Banach Space. The difficulty is to reconcile the conflicting requirements of simplicity, precision, and usefulness in the sense that analogues of certain classical theorems apply. J.-P. Aubin discusses one such candidate, that provided by Ioffe's 'fans', describes its properties and contrasts it with alternatives. I. Ekeland motivates and proves in a special case a 'mean value' theorem due to Ioffe, given in terms of Clarke's generalized derivatives, governing locally Lipschitzian functions on $\mathbb{R}^n$.

Questions concerning the existence of measurable selections of multivalued mappings are intimately connected with the study of both the existence and the characterization of solutions to optimization problems. J.E. Jayne and C.A. Rogers prove a new measurable selection theorem and apply it to obtain generalizations of a theorem by J. Saint Raymond on the invariance of Borel classes.

The papers by L.C. Young and R. B. Vinter illustrate the role of convexity even in optimization and modelling problems which are not 'convex' in a conventional sense. The notions of 'generalized curves' and 'generalized flows' provide the machinery for viewing controls as defining linear functionals and for determining global optimality conditions for general optimal control problems in terms of subdifferentials of convex functions. R.B. Vinter presents contributions of A.D. Ioffe concerning such global conditions in the perspective given by subsequent work, states some new results and speculates on connections with local necessary conditions. L.C. Young argues that generalized curves are not merely convenient mathematical devices for proving theorems in optimal control theory, but reflect fundamental physical phenomena. In his paper he describes the physical concepts which generalized curves expose and gives a preview of Nowasad's theory concerning their applications to the modelling of elementary particles.

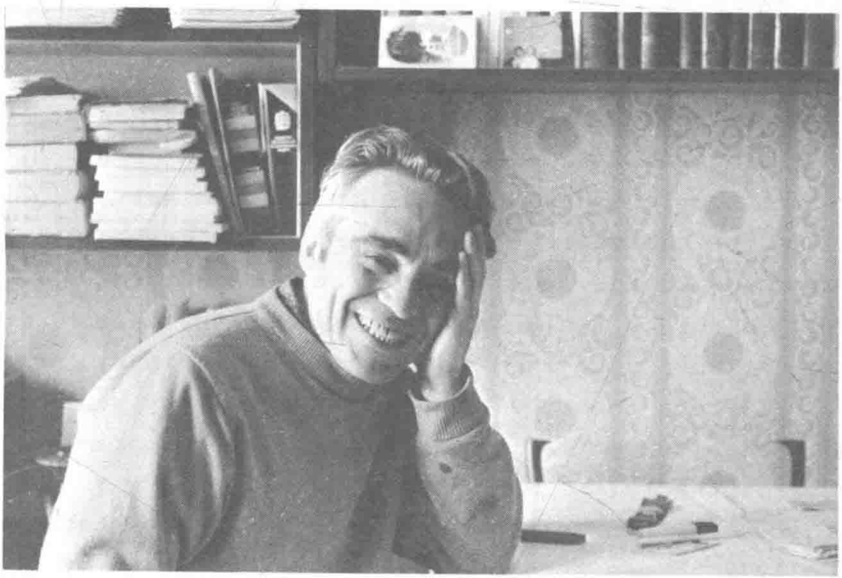
The papers by J.-B. Hiriart-Urruty and J.-P. Crouzeix treat a generalization of the notion of the subdifferential of a convex function, namely the ε-subdifferential and, in quasi-convexity, a generalization of the notion of convexity itself. J.-B. Hiriart-Urruty surveys some known, and proves some new, results on ε-differentials of composite functions. J.-P. Crouzeix provides a state of the art review of the continuity and differentiability properties of quasi-convex functions on $\mathbb{R}^n$ and proves a new theorem concerning almost everywhere differentiability of (possibly discontinuous) quasi-convex functions.

The collection also contains a paper by A.D. Ioffe. Here, a more detailed discussion than that appearing in J.-P. Aubin's paper is given of certain aspects of the theory of fans and of its relevance to optimization theory.

Professor Ioffe was prevented from taking up his invitation to the Colloquium. The paper under his name, which takes the place of the paper he was unable to present in person, is due to be published in the Annals of the New York Academy of Sciences. It appears here also, by kind permission of Professor J. Liebowitz.

We gratefully acknowledge help we received from many people in organizing the Colloquium and preparing the proceedings. In particular, we should like to thank Professor G.E.H. Reuter, head of the Mathematics Department and Professor M.L. Lehman, head of the Computing Department, Imperial College, for making available to us the facilities of their departments. We offer a special word of thanks too to Mrs Doris Abeysekera for her excellent job in typing the manuscript.

July 1981 J.-P. Aubin and R.B. Vinter



Alexander D. Ioffe

Dedication

This collection of papers is dedicated to Professor Alexander D. Ioffe in acknowledgement of his substantial contributions to optimization theory and related branches of analysis.

In the 1960's, a period of great advances in convex analysis, Ioffe was active in this area and its applications in optimization theory. His work on Orlitz-type spaces, convex duality, detailed characterization of subdifferentials and applications to minimization problems with integral constraints dates from this time. Subsequently he did important work in the calculus of variations and optimal control theory on existence of solutions and Carathéodory-type sufficient conditions in Optimal Control Theory. He co-authored with V.M. Tihomirov an expository text 'Theory of Extremum Problems' which was published in English translation by North Holland in 1979.

His recent contributions have been in descriptive set theory and non-smooth analysis. In non-smooth analysis he has developed an approach to the local analysis of non-smooth functions taking values in finite dimensional Banach spaces in which approximation is provided by set valued mappings, called 'fans'. Ioffe's theory of fans has led to very general necessary conditions for optimality in mathematical programming and optimal control theory.

Ioffe obtained a PhD degree in Mathematics at Moscow State University. In 1972 he was appointed Associate Professor of Applied Mathematics at the Moscow Technological Institute for High-Speedway Construction.

He is still active as a researcher; perhaps, even, is doing his best work now. He writes: "the last decade has been a happy time for optimization theory. Until then the main effort of researchers in the field was concentrated on studying nonclassical problems using mostly classical methods.

Now the very foundations of analysis are undergoing substantial revision. I have a feeling that soon the scaffolding will be removed and we shall see the edifice of a new theory comparable in beauty and strength to the classical variational calculus but much more spacious. I would be happy to be able to add something to the construction of this edifice. This is my main internal stimulus".

The many friends and colleagues of this fine mathematician, and kind and generous man, wish him well. We express the sincere hope that he will continue to make valuable contributions to construction of the 'edifice'.

Jean-Pierre Aubin and Richard Vinter

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J-P AUBIN

Ioffe's fans and generalized derivatives of vector-valued maps

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1. INTRODUCTION

We put in perspective A.D. Ioffe's approach for defining generalized derivatives of locally Lipschitz vector-valued maps. For that purpose, we recall some of the numerous attempts to generalize the concept of derivatives of real-valued functions: Schwartz's derivatives in the distributional sense, Dini's and Clarke's generalized derivatives and contingent derivatives. Then, we mention how the problem of devising a generalized differential calculus for vector-valued maps can be tackled : it is at this point that we introduce Ioffe's approach and compare it with Clarke's concept of generalized Jacobian and the derivatives defined through tangent cones to the graph. Ioffe's candidate for a generalized derivative is a positively homogeneous set-valued map, with closed convex values, sub-additive in some sense. It is called a "fan" and enjoys many properties analogous to the properties of convex processes (set valued maps whose graphs are closed convex cones). We end this communication with a presentation of Ioffe's fans.

2. GENERALIZED DERIVATIVES OF REAL-TIME AND FUNCTIONS

Since the times when Newton and Leibnitz laid the foundations of differential calculus, many mathematicians have attempted to differentiate functions that are not differentiable in the usual sense, for derivatives take into account the physical notion of variation of phenomena and thus play a crucial role in models of our environment, in particular, in those whose aim is to explain the evolution of a system.

These problems are still in the process of being studied and new problems of this type arise continuously. Quite early on, the technical definition of a derivative appeared to be too restrictive. The mathematical idea - instantaneous variation- lying behind the concept of differentiation ought to be retained, while its technical translation should be more flexible to permit adaptation to the problem at hand.

As a first important example, let us mention the concept of *derivative of a distribution* discovered by Laurent Schwartz (see Laurent Schwartz [19]).

Let f be a locally integrable function defined on an open subset $\Omega \in \mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be a chosen direction. We form the *differential quotient*

$$\nabla_h f(.,v) = \frac{f(.+hv) - f(.)}{h} \tag{2.1}$$

Instead of requiring that the functions $\nabla_h f(.,v)$ converge in the topology of the *pointwise convergence*, one is still content with the much weaker convergence of $\nabla_h f(.,v)$ in the *space of distributions*. Furthermore, one can tune the concept of generalized derivative by making precise the strongest topology for which the convergence of $\nabla_h f(.,v)$ holds. Anyone knows the importance of distributions in both pure and applied analysis.

However, many problems arising in non-linear analysis, in optimization and in stability theory for differential equations still require some kind of pointwise convergence of the differential quotients $\nabla_h f(.,v)$ but allow us to use lim sup or lim inf instead of the limit. This was already proposed by Dini when F is a locally Lipschitz function : for instance, he proposed the use of

$$\liminf_{h \to o^+} \nabla_h f(x,v) \, . \tag{2.2}$$

A few years ago, Clarke [5] suggested the use of the *Clarke generalized derivative* defined by

$$D_C f(x)(v) = \limsup_{\substack{h \to o^+ \\ y \to x}} \nabla_h f(y,v) \tag{2.3}$$

whose charm lies in the fact that it is always *convex* and *continuous* with respect to v. More recently, Clarke [5] and Rockafellar [18] proposed a quite involved definition valid for any function, which coincides with Clarke's derivative when F is locally Lipschitz.

Also, we proposed in [4] consideration of another candidate, called the *contingent derivative*, defined by

$$\liminf_{\substack{h \to o^+ \\ w \to v}} \nabla_h f(x,w) \tag{2.4}$$

which is quite well adapted to studying problems arising in non-linear analysis and differential equations. So, the problem of studying workable concepts of generalized derivatives of real-valued functions is quite well grounded.

3. GENERALIZED DERIVATIVES OF VECTOR-VALUED FUNCTIONS

What about vector-valued functions F from $\Omega \in \mathbb{R}^n$ to a vector space $\mathbb{R}^p$? The situation is perhaps more difficult and many concepts have been proposed, among them, Ioffe's fans.

(a) In the framework of distributions, Laurent Schwartz [19] defined vector-valued functions and their derivatives.

(b) In [6], Clarke introduced the concept of *generalized Jacobian* of a locally Lipschitz function F at x, which is a *closed convex subset* $\partial F(x)$ of matrices from $\mathbb{R}^n$ to $\mathbb{R}^p$. Namely, $\partial F(x)$ is the convex hull of all matrices of the form $M := \lim_{y_n \to x} \nabla F(y_n)$ where y_n converges to x and F is differentiable at y_n for each n. This is possible thanks to Rademacher's theorem, which states that a locally Lipschitz function F is almost everywhere differentiable. When F is continuously differentiable, $\nabla F(x)$ is the only element of $\partial F(x)$. When F is a real-valued function, $\partial F(x)$ reduces to the Clarke generalized gradient.

(c) Ioffe suggested in [12] another approach. Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ be a locally Lipschitz map. The idea is the following. He associates with a map F from $\Omega \subset \mathbb{R}^n$ to $\mathbb{R}^p$ the *linear operator* $F^\star$ from $\mathbb{R}^{p\star}$, dual of $\mathbb{R}^p$, to $LL(\Omega)$, the space of locally Lipschitz real-valued functions on Ω defined by

$$\forall y^\star \in \mathbb{R}^{p^\star},\ F^\star y^\star : x \in \Omega \longmapsto F^\star y^\star(x) := \langle y^\star, F(x)\rangle. \quad (3.1)$$

Since for any $y^\star \in \mathbb{R}^{p^\star}$, $F^\star y^\star$ is a locally Lipschitz function, we can use its Clarke generalized derivative, defined by

$$dF(x;y^*,v) := D_C(F^*y^*)(x,v) \tag{3.2}$$

The properties of the Clarke derivatives (see Clarke [5], [7], or [3]) imply that

$$\begin{cases} \text{the maps } y^* \longmapsto dF(x;y^*,v) \text{ and } v \longrightarrow dF(x;y^*,v) \text{ are} \\ \textit{convex and positively homogeneous.} \end{cases} \tag{3.3}$$

Furthermore $v \longmapsto dF(x;y^*,v)$ is continuous :

$$\exists\ c > 0 \text{ such that } |dF(x,y^*,v)| \leq c\| v \| \tag{3.4}$$

and we have this property of symmetry :

$$dF(x,y^*,-v) = dF(x,-y^*,v) \ . \tag{3.5}$$

Taking this function $dF(x;.,\cdot)$ we associate with it a subset $\bar{\partial}F(x)$ of matrices M from $\mathbb{R}^n$ to $\mathbb{R}^p$ defined by

$$M \in \bar{\partial}F(x) \text{ if and only if } \forall\ v \in \mathbb{R}^n,\ \forall y^* \in \mathbb{R}^{p^*}, \tag{3.6}$$

$$\langle y^*,Mv\rangle \leq dF(x;y^*,v).$$

The definition of the Clarke's generalized Jacobian $\partial F(x)$ shows that :

$$\partial F(x) \subset \bar{\partial}F(x). \tag{3.7}$$

It is not known whether $\partial F(x) = \bar{\partial}F(x)$ or not , or, when X and Y are infinite dimensional spaces, whether we can prove that $\bar{\partial}F(x)$ is nonempty. Also, the conditions under which

$$\sup_{M \in \bar{\partial}F(x)} \langle y^*,Mv\rangle = dF(x;y^*,v) \tag{3.8}$$