AN INTRODUCTION TO THE STUDY OF INTEGRAL EQUATIONS

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PREFACE

In this tract I have tried to present the main portions of the theory of integral equations in a readable and, at the same time, accurate form, following roughly the lines of historical development. I hope that it will be found to furnish the careful student with a firm foundation which will serve adequately as a point of departure for further work in this subject and its applications. At the same time it is believed that the legitimate demands of the more superficial reader, who seeks results rather than proofs, will be satisfied by the precise statement of these results as italicised, and therefore easily recognized, theorems. The index has been added to facilitate the use of the booklet as a work of reference.

In these days of rapidly multiplying voluminous treatises, I hope that the brevity of this treatment may prove attractive in spite of the lack of exhaustiveness which such brevity necessarily entails if the treatment, so far as it goes, is to be adequate.

I wish to thank Professor Max Mason of the University of Wisconsin who has helped me with some valuable criticisms; and I shall be grateful to any readers who may point out to me such errors as still remain.

MAXIME BÔCHER.

Harvard University,
Cambridge, Mass.
November, 1908.

This second edition is a reprint of the first, in which, however, such errors as have come to my notice have been corrected. Of these the most serious (on pages 17 and 62–64) were called to my attention by Professor D. R. Curtiss and Dr W. A. Hurwitz respectively.

M. B.

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AN INTRODUCTION TO THE STUDY OF INTEGRAL EQUATIONS

Introduction. The theory and applications of integral equations, or, as it is often called, of the inversion of definite integrals, have come suddenly into prominence and have held during the last half dozen years a central place in the attention of mathematicians. By an integral equation* is understood an equation in which the unknown function occurs under one or more signs of definite integration. Mathematicians have so far devoted their attention mainly to two peculiarly simple types of integral equations,—the linear equations of the first and second kinds,—and we shall not in this tract attempt to go beyond these cases. We shall also restrict ourselves to equations in which only simple (as distinguished from multiple) integrals occur. This restriction, however, is quite an unessential one made solely to avoid unprofitable complications at the start, since the results we shall obtain usually admit of an obvious extension to the case of multiple integrals without the introduction of any new difficulties†. In this respect integral equations are in striking contrast to the closely related differential equations, where the passage from ordinary to partial differential equations is attended with very serious complications.

The theory of integral equations may be regarded as dating back at least as far as the discovery by Fourier of the theorem concerning integrals which bears his name; for, though this was not the point of view of Fourier, this theorem may be regarded as a statement of the solution of a certain integral equation of the first kind‡. Abel and Liouville, however, and after them others began the treatment of special integral equations in a perfectly conscious way, and many of them perceived clearly what an important place the theory was destined to fill§.

* The term Integral Equation was suggested by du Bois-Reymond. Cf. *Crelle*, vol. 103 (1888), p. 228.

† Another extension, in which serious complications do not usually arise, is to *systems* of integral equations. We do not consider such systems in this Tract.

‡ Cf. the closing page of this Tract.

§ Cf., besides the article of du Bois-Reymond already cited, some remarks by Rouché, Paris *C. R.* vol. 51 (1860), p. 126.

As we shall not, except in one relatively unimportant case, take up any of the applications of the subject, it may be well to say explicitly that like so many other branches of analysis the theory was called into being by specific problems in mechanics and mathematical physics. This was true not merely in the early days of Abel and Liouville, but also more recently in the cases of Volterra and Fredholm. Such applications of the theory, together with its relations to other branches of analysis*, are what give the subject its great importance.

1. Some Preliminary Propositions and Definitions. In order to avoid interruptions in later sections, we collect here certain propositions of the integral calculus for future reference.

We shall have to deal with functions of one and of two variables. The independent variables, which we will for the present denote by x and (x, y) respectively, are in all cases real. In fact, in order to avoid unnecessary complications we will assume that, unless the contrary is explicitly stated, *all quantities we have to deal with are real.*

The range of values of the single argument x is usually

$$I \qquad a \leqq x \leqq b.$$

We shall speak of this in future simply as the interval I.

In the case of functions of two variables, two cases have to be considered. Interpreting (x, y) as rectangular coordinates in a plane, we sometimes consider the square

$$S \qquad \begin{Bmatrix} a \leqq x \leqq b \\ a \leqq y \leqq b \end{Bmatrix},$$

and sometimes the triangle

$$T \qquad a \leqq y \leqq x \leqq b.$$

It should be noticed that the three regions we have just defined, I, S, T, are closed regions, that is they include the points of their boundaries.

In order to avoid long circumlocutions we lay down the following

DEFINITION. *We say that the discontinuities of a function of* (x, y) *are regularly distributed in* S *or in* T *if they all lie on a finite number of curves with continuously turning tangents no one of which is met by a line parallel to the axis of* x *or of* y *in more than a finite number of points.*

In order to make the enunciation of some of our results simpler, we will assume once for all that the functions we deal with are defined even

* Cf., for instance, much of Hilbert's work.

at the points of discontinuity, at least in the cases where they remain finite in the neighbourhood of such points.

The following theorem will be important for us. We state it first for the case of the region S.

THEOREM 1. *If the two functions $\phi(x, y)$ and $\psi(x, y)$ are finite in S and their discontinuities, if they have any, are regularly distributed, the function*

$$F(x, y) = \int_a^b \phi(x, \xi)\, \psi(\xi, y)\, d\xi$$

is continuous throughout S.

The truth of this theorem becomes evident if we interpret (x, y, ξ) as rectangular coordinates in space. It is then clear that the function under the integral sign is finite throughout the cube

$$a \leqq x \leqq b, \quad a \leqq y \leqq b, \quad a \leqq \xi \leqq b,$$

and becomes discontinuous in this cube only at points on two sets of cylinders whose generators are parallel respectively to the axes of x and y. Moreover these cylinders are so shaped that any line $x = x_0$, $y = y_0$ in this cube meets them at only a finite number of points.—The formal proof, based on these or similar considerations, presents no difficulty, and we leave it for the reader.

COROLLARY. *If $\phi(x, y)$ and $\psi(x, y)$ are finite in T and their discontinuities, if they have any, are regularly distributed, the function*

$$H(x, y) = \int_y^x \phi(x, \xi)\, \psi(\xi, y)\, d\xi$$

is continuous throughout T.

This is merely a special case of Theorem 1. For if we define ϕ and ψ to have the value zero everywhere outside of T, it is clear that they satisfy the conditions of Theorem 1 throughout S and that the function $F(x, y)$ reduces to $H(x, y)$.

If $\phi(x, y)$ satisfies the conditions of Theorem 1, the double integral of ϕ extended over S may be evaluated in either one or two ways as an iterated integral* and we thus get the formula

$$\int_a^b \int_a^b \phi(x, y)\, dy\, dx = \int_a^b \int_a^b \phi(x, y)\, dx\, dy.$$

If, in particular, ϕ vanishes everywhere outside of T, we get

* By a double integral we understand the limit of a sum obtained by dividing up the region in question into pieces both of whose dimensions are small. By an iterated integral, the integral of an integral.

DIRICHLET'S FORMULA*. *If ϕ is finite in T and its discontinuities, if it has any, are regularly distributed, then*

$$\int_a^b\int_a^x \phi(x, y)\,dy\,dx = \int_a^b\int_y^b \phi(x, y)\,dx\,dy.$$

This formula admits of extension to certain cases in which the integrand does not remain finite in T. The most general case of this sort which we shall have occasion to use is contained in the following statement, for a simple proof of which we refer to the first part of a paper by W. A. Hurwitz†:

DIRICHLET'S EXTENDED FORMULA. *If $\phi(x, y)$ is finite in T and its discontinuities, if it has any, are regularly distributed, and if λ, μ, ν are constants such that*

$$0 \leqq \lambda < 1, \quad 0 \leqq \mu < 1, \quad 0 \leqq \nu < 1,$$

then

$$\int_a^b\int_a^x \frac{\phi(x, y)\,dy\,dx}{(x-y)^\lambda (b-x)^\mu (y-a)^\nu} = \int_a^b\int_y^b \frac{\phi(x, y)\,dx\,dy}{(x-y)^\lambda (b-x)^\mu (y-a)^\nu}.$$

Finally we turn to some theorems concerning functions of a single variable.

THEOREM 2. *If $\phi(x)$ is finite and has only a finite number of discontinuities in I, the function*

$$\Phi(x) = \int_a^x \frac{\phi(\xi)\,d\xi}{(x-\xi)^\lambda} \qquad (\lambda < 1)$$

is continuous throughout I, including the point a, where it vanishes‡.

To prove this we introduce the new variable of integration

$$s = \frac{\xi - a}{x - a}.$$

Then

$$\Phi(x) = (x-a)^{1-\lambda}\int_0^1 \frac{\phi[a + s(x-a)]}{(1-s)^\lambda}\,ds = (x-a)^{1-\lambda}\,\Psi(x) \qquad (a < x \leqq b).$$

* Cf. *Crelle's Journal*, vol. 17 (1837), p. 45.

† *Annals of Mathematics*, vol. 9 (1908), p. 183. This result may also be deduced from a general theorem of de la Vallée Poussin. Cf. the *Cours d'Analyse* of this author, vol. 2, pp. 89—95.

‡ We *define* the symbol $\int_a^a \psi(x)\,dx$ to mean zero, whatever the nature of the function ψ may be.

By replacing ϕ by the upper limit of its absolute value, we see that $\Psi(x)$ remains finite, and hence that Φ approaches zero as x approaches a. Consequently Φ is continuous at a. On the other hand the same substitution shows that the integral Ψ converges uniformly when $a < x \leqq b$. For any fixed positive $\delta < 1$ the function

$$\Psi_1(x) = \int_0^{1-\delta} \frac{\phi[a + s(x-a)]}{(1-s)^\lambda} ds$$

is continuous throughout the interval $a < x \leqq b$, since the integrand in Ψ_1 is finite in the rectangle

$$a < x \leqq b, \quad 0 \leqq s \leqq 1-\delta,$$

and is discontinuous only along a finite number of curves in this rectangle each of which is met by a line $x = x_0$ in at most one point. Since, as we have just seen, $\Psi_1(x)$ approaches $\Psi(x)$ uniformly as δ approaches zero, it follows from a fundamental theorem in uniform convergence that $\Psi(x)$ is continuous when $a < x \leqq b$, and hence the same is true of $\Phi(x)$, and our theorem is proved.

THEOREM 3. *If, in I, $\phi(x)$ is continuous and has a derivative which is finite and which has at most a finite number of discontinuities in I, and if $\phi(a) = 0$, the function*

$$\Phi(x) = \int_a^x \frac{\phi(\xi)}{(x-\xi)^\lambda} d\xi \qquad (\lambda < 1)$$

has a derivative continuous throughout I and given by the formula

$$\Phi'(x) = \int_a^x \frac{\phi'(\xi)}{(x-\xi)^\lambda} d\xi.$$

For if we integrate the expression for $\Phi(x)$ by parts, we have, when we remember that $\phi(a) = 0$,

$$\Phi(x) = \frac{1}{1-\lambda} \int_a^x (x-\xi)^{1-\lambda} \phi'(\xi) d\xi.$$

Applying here the rule for differentiating an integral whose limits are variable, we get the desired expression for $\Phi'(x)$. Hence from Theorem 2 it is evident that $\Phi'(x)$ is continuous. It should be noticed that when $\lambda > 0$ the integrals with which we have to deal are *infinite integrals* (i.e. integrals in which the integrand does not remain finite) so that the application to them of the ordinary rules of the calculus requires careful justification.

An alternative form of proof for this theorem consists in applying Dirichlet's Extended Formula* as follows:

$$\Phi(x) = \int_a^x \frac{1}{(x-\xi)^\lambda} \int_a^\xi \phi'(s)\,ds\,d\xi = \int_a^x \phi'(s) \int_s^x \frac{d\xi}{(x-\xi)^\lambda}\,ds$$
$$= \int_a^x \phi'(s) \int_s^x \frac{d\xi}{(\xi-s)^\lambda}\,ds = \int_a^x \int_a^\xi \frac{\phi'(s)}{(\xi-s)^\lambda}\,ds\,d\xi.$$

The differentiation of this last formula gives us the result we wish to establish†.

In conclusion we point out by means of the following two examples that if we replace the condition of finiteness for ϕ' by the condition of integrability, or even of absolute integrability, Φ will not always have a continuous derivative:

(1) $\phi(x) = (x-a)^\lambda$, $\quad \Phi(x) = k(x-a)$,

(2) $\phi(x) = \begin{cases} 0 & (a \leqq x \leqq a') \\ (x-a')^\lambda & (a' < x \leqq b) \end{cases}$, $\quad \Phi(x) = \begin{cases} 0 & (a \leqq x \leqq a') \\ k(x-a') & (a' < x \leqq b) \end{cases}$.

In both cases k is a positive constant, and if $0 < \lambda < 1$, ϕ is continuous in I and has a derivative which is continuous except at one point and absolutely integrable but not finite. In the first case Φ' is continuous, in the second discontinuous.

2. Abel's Mechanical Problem. In one of his earliest published papers‡ Abel showed how a certain mechanical problem, which includes the problem of the tautochrone as a special case, leads to what has since come to be called an integral equation, on whose solution the solution of the problem depends. On account of its great historical interest, we take up this problem in this section.

A particle starting from rest at a point P on a smooth curve which lies in a vertical plane, slides down the curve to its lowest point O. The velocity

* It should be noticed that we use this formula here under slightly different restrictions on the function $\phi(x, y)$ since ϕ is now a function of y alone, and therefore if it is discontinuous at all, is discontinuous along lines parallel to the axis of x.

† This method of reasoning admits of immediate extension to the proof of the more general formula

$$\frac{d}{dx}\int_a^x \psi(x-\xi)\,\phi(\xi)\,d\xi = \int_a^x \psi(x-\xi)\,\phi'(\xi)\,d\xi,$$

which holds under suitable restrictions on ψ.

‡ See *Collected Works*, p. 11. This paper was first published in Christiania in 1823. Cf. also a second paper beginning on p. 97 of the *Collected Works*, and originally published in *Crelle*, vol. 1 (1826), p. 153.

acquired at O will be independent of the shape of the curve. The time of descent T will however depend on this shape. We take O as origin, the axis of x vertically upward, and the axis of y horizontal and in the plane of the curve. Let the coordinates of the point of departure P be (x, y), and the coordinates of the point Q reached by the particle at the time t be (ξ, η), g the gravitational constant, and s the arc OQ. The velocity of the particle at Q is

$$-\frac{ds}{dt} = \sqrt{2g\,(x-\xi)}.$$

Hence
$$\sqrt{2g}\,t = -\int_P^Q \frac{ds}{\sqrt{x-\xi}}.$$

If we express s in terms of ξ

$$s = v(\xi),$$

the whole time of descent is then

$$T = \frac{1}{\sqrt{2g}} \int_0^x \frac{v'(\xi)\,d\xi}{\sqrt{x-\xi}}.$$

If the shape of the curve is given, the function v may be computed, and the whole time of descent is given to us as a function of x by the last formula.

The problem which Abel set himself is the converse of this, namely to determine the curve for which the time of descent is a given function of x. If we write

$$\sqrt{2g}\,T = f(x),$$

our problem is to determine the function v from the equation

$$f(x) = \int_0^x \frac{v'(\xi)\,d\xi}{\sqrt{x-\xi}} \qquad (1).$$

The formula for the solution of this integral equation was obtained by Abel by two different methods. The first depends on the use of series proceeding according to powers, not necessarily integral, of the argument; while the second, of a more general character, is closely related to the one we are about to give in the next section. Neither of Abel's methods can be regarded as satisfactory although they lead to the correct result. Among other objections it may be said that both methods omit the essential step of proving that the equation (1) has a solution.

3. Solution of Abel's Equation*. Instead of the equation (1) of § 2, Abel set himself the problem of solving a more general equation which we will write in the form

$$f(x) = \int_a^x \frac{u(\xi)\,d\xi}{(x-\xi)^\lambda} \qquad (0 < \lambda < 1) \qquad (1),$$

where f is a known function, u the function to be determined.

In order to solve (1) we begin by establishing the general formula (2) below. We start from the well-known formula

$$\frac{\pi}{\sin \mu\pi} = \int_\xi^z \frac{dx}{(z-x)^{1-\mu}(x-\xi)^\mu} \qquad (0 < \mu < 1).$$

Let $\phi(\xi)$ be any function which is continuous and has a continuous derivative throughout I. Multiply this equation by $\phi'(\xi)\,d\xi$ and integrate from a to z, which we suppose to be any point of I. This gives

$$\frac{\pi}{\sin \mu\pi}[\phi(z) - \phi(a)] = \int_a^z \int_\xi^z \frac{\phi'(\xi)}{(z-x)^{1-\mu}(x-\xi)^\mu}\,dx\,d\xi.$$

If we apply Dirichlet's Generalized Formula to the second member of this equation, we get the desired result

$$\phi(z) - \phi(a) = \frac{\sin \mu\pi}{\pi} \int_a^z \frac{1}{(z-x)^{1-\mu}} \int_a^x \frac{\phi'(\xi)\,d\xi}{(x-\xi)^\mu}\,dx \qquad (2),$$

a formula which holds under the sole restrictions that z be in I, and ϕ be continuous and have a continuous derivative in I, and that

$$0 < \mu < 1.$$

Theorem 2, § 1, shows us at once that a necessary condition that (1) have a solution continuous throughout I is that $f(x)$ be continuous throughout I and that $f(a) = 0$.

Let us suppose that these conditions are fulfilled and that $u(x)$ is a continuous solution of (1). Multiply (1) by $(z-x)^{\lambda-1}\,dx$, where z is a point of I, and integrate from a to z, thus getting

$$\int_a^z \frac{f(x)\,dx}{(z-x)^{1-\lambda}} = \int_a^z \frac{1}{(z-x)^{1-\lambda}} \int_a^x \frac{u(\xi)\,d\xi}{(x-\xi)^\lambda}\,dx.$$

If in (2) we let

$$\phi(x) = \int_a^x u(\xi)\,d\xi,$$

* Except for the method of deducing formula (2), the method we use is, barring notation, that of Liouville in *Liouville's Journal*, vol. 4 (1839), p. 233. Liouville, who seems not to have been aware of Abel's work, had already published on this subject in the *Journal de l'École Polytechnique*, Cahier 21 (1832), p. 1.

it will be seen that the preceding equation reduces to

$$\int_a^z \frac{f(x)\,dx}{(z-x)^{1-\lambda}} = \frac{\pi}{\sin\lambda\pi}\int_a^z u(\xi)\,d\xi \qquad (3).$$

Since the second member of (3) has a continuous derivative with regard to z, the same must be true of the first member, and this gives us a further necessary condition for (1) having a continuous solution. By differentiating (3), we get as the value of this solution

$$u(z) = \frac{\sin\lambda\pi}{\pi}\frac{d}{dz}\int_a^z \frac{f(x)\,dx}{(z-x)^{1-\lambda}} \qquad (4).$$

We thus see that u is completely determined, that is that (1) does not have more than one continuous solution. That the formula (4) really does give a solution of (1) may be seen by substituting it in (1). The second member of (1) thus becomes

$$\frac{\sin\lambda\pi}{\pi}\int_a^x \frac{1}{(x-\xi)^\lambda}\frac{d}{d\xi}\int_a^\xi \frac{f(x)\,dx}{(\xi-x)^{1-\lambda}}\,d\xi,$$

which reduces by means of Theorem 3, § 1, to

$$\frac{\sin\lambda\pi}{\pi}\frac{d}{dx}\int_a^x \frac{1}{(x-\xi)^\lambda}\int_a^\xi \frac{f(x)\,dx}{(\xi-x)^{1-\lambda}}\,d\xi,$$

and this in turn reduces by means of (2), when we let

$$\phi(z) = \int_a^z f(x)\,dx,$$

to

$$\frac{d}{dx}\int_a^x f(z)\,dz = f(x).$$

Thus we see that (4) is a solution of (1), and we have proved

THEOREM 1. *A necessary and sufficient condition that* (1) *have a solution continuous in I is that $f(x)$ be continuous in I, that $f(a)=0$, and that*

$$\int_a^x \frac{f(\xi)\,d\xi}{(x-\xi)^{1-\lambda}}$$

have a continuous derivative throughout I. If these conditions are fulfilled, (1) *has only one continuous solution, given by formula* (4).

An important case in which these conditions are fulfilled is that in which f is continuous and has a derivative which is finite, and has at most a finite number of discontinuities in I, and $f(a)=0$. This we see from Theorem 3, § 1, from which we also see that in this case (4) may be written

$$u(z) = \frac{\sin\lambda\pi}{\pi}\int_a^z \frac{f'(x)\,dx}{(z-x)^{1-\lambda}} \qquad (5).$$

Hence

THEOREM 2. *If $f(x)$ is continuous and has a derivative finite in I and with only a finite number of discontinuities there, and $f(a)=0$, equation* (1) *has one and only one continuous solution, and this is given by formula* (5) *.

While this is essentially Abel's result, that mathematician did not consider the integral equation (1) but rather the differentio-integral equation

$$f(x)=\int_a^x \frac{v'(\xi)\,d\xi}{(x-\xi)^\lambda} \qquad (0<\lambda<1) \qquad (6).$$

By means of the theorems just established and Theorems 2, 3 of § 1 we readily deduce the result

THEOREM 3. *A necessary and sufficient condition that* (6) *have a solution which together with its derivative is continuous throughout I is that $f(x)$ be continuous in I, that $f(a)=0$, and that*

$$\int_a^x \frac{f(\xi)\,d\xi}{(x-\xi)^{1-\lambda}}$$

have a continuous derivative throughout I. If these conditions are fulfilled, the general solution of (6) *is*

$$v(z)=k+\frac{\sin\lambda\pi}{\pi}\int_a^z \frac{f(x)\,dx}{(z-x)^{1-\lambda}},$$

where k is an arbitrary constant.

By letting $\lambda=\frac{1}{2}$ we get the solution of the mechanical problem of § 2. If in particular we let $f(x)=\text{const.}$, we get Abel's solution of the problem of the tautochrone.

An easy extension of the results we have found is to the case in which

* Goursat, in *Acta Math.* vol. 27 (1903), pp. 131—133, has shown that equation (1) still has a solution, though not a continuous one, if we drop the requirement that $f(a)=0$. This may be readily seen by bringing in, in place of u, the function

$$v(x)=u(x)-\frac{\sin\lambda\pi}{\pi}\frac{f(a)}{(x-a)^{1-\lambda}}.$$

Making this substitution, we find that equation (1) reduces to

$$f(x)-f(a)=\int_a^x \frac{v(\xi)\,d\xi}{(x-\xi)^\lambda}.$$

Conversely, we see that a solution of this last equation corresponds to a solution of (1). Consequently a solution of (1) is

$$u(z)=\frac{\sin\lambda\pi}{\pi}\frac{f(a)}{(z-a)^{1-\lambda}}+\frac{\sin\lambda\pi}{\pi}\int_a^z \frac{f'(x)\,dx}{(z-x)^{1-\lambda}}.$$